

## ON GENERALIZED FUZZY MULTISSETS AND THEIR USE IN COMPUTATION

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ABSTRACT. An orthogonal approach to the fuzzification of both multisets and hybrid sets is presented. In particular, we introduce  $L$ -multi-fuzzy and  $L$ -fuzzy hybrid sets, which are general enough and in spirit with the basic concepts of fuzzy set theory. In addition, we study the properties of these structures. Also, the usefulness of these structures is examined in the framework of mechanical multiset processing. More specifically, we introduce a variant of fuzzy P systems and, since simple fuzzy membrane systems have been introduced elsewhere, we simply extend previously stated results and ideas.

### 1. Introduction

Intuitively, a set is a collection of elements (e.g., numbers or symbols) that is completely determined by them.<sup>1</sup> The elements of a set are pairwise different. If we relax this restriction and allow repeated occurrences of any element, then we end up with a mathematical structure that is known as *multiset*<sup>2</sup> (see [5] for a historical account of the development of the multiset theory; also, see [15] for a recent account of the mathematical theory of multisets). Multisets are really useful structures and they have found numerous applications in mathematics and computer science. For example, the prime factorization of an integer  $n > 0$  is a multiset  $\mathcal{N}$  whose elements are primes. Also, every monic polynomial  $f(x)$  over the complex numbers corresponds in a natural way to the multiset  $\mathcal{F}$  of its roots. In addition, multisets have been used in concurrency theory [6]. A rather interesting recent development in the theory of multisets is the discovery that the logic of multisets is the  $\{\otimes, \multimap, \oplus, \mathbf{1}\}$ -fragment of intuitionistic linear logic (see [19, 20] for details).

If we allow elements of a multiset to occur an integral number of times (and that includes a *negative* number of times), we end up with a structure that has been dubbed *hybrid set*. These structures have been introduced by Loeb [9]. Initially, one may wonder whether hybrid sets are of any use. However, Loeb has shown that they are indeed very useful structures (see [9, 2]). For example, one can use a hybrid set to describe the roots and the poles of a rational function. In particular,

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<sup>1</sup>For the present discussion this vague definition is adequate, but it may lead to paradoxes like the “set of all sets” paradox, which is known in the literature as Russell’s paradox. However, such paradoxes will not concern us here.

<sup>2</sup>The term “multiset” has been coined by N.G. de Bruijn [8].

if  $f(x)$  is a monic rational function, then  $f(x)$  can be written in terms of its roots  $a_1, a_2, \dots, a_n$ , and its poles  $b_1, b_2, \dots, b_m$ , as follows:

$$f(x) = c \frac{(x - a_1)(x - a_2) \cdots (x - a_n)}{(x - b_1)(x - b_2) \cdots (x - b_m)}$$

From this we can directly form a hybrid set, where elements that occur a positive number of times correspond to the roots of the function and elements that occur a negative number of times correspond to the poles of the function.

In a seminal paper, Yager [23] introduced fuzzy multisets, that is fuzzy subsets whose elements may occur more than one time (see [11] for an up-to-date presentation of the theory of fuzzy multisets, which, however, does not differ significantly from [10]). Yager defined fuzzy multisets as follows [23]:

**Definition 1.1.** Assume  $X$  is a set of elements. Then a fuzzy bag<sup>3</sup>  $A$  drawn from  $X$  can be characterized by a function  $\text{Count.Mem}_A$  such that

$$\text{Count.Mem}_A : X \rightarrow Q,$$

where  $Q$  is the set of all crisp bags drawn from the unit interval.

In “modern parlance” fuzzy multisets can be characterized by a high-order function. In particular, a fuzzy multiset  $A$  can be characterized by a function

$$A : X \rightarrow \mathbb{N}^I,$$

where  $I = [0, 1]$  and  $\mathbb{N}$  is the set of natural numbers including zero. It is not difficult to see that any fuzzy multiset  $A$  is actually characterized by a function

$$A : X \times I \rightarrow \mathbb{N},$$

which is obtained from the former function by *uncurrying* it. However, it is more natural to demand that for each element  $x$  there is only one membership degree and one multiplicity. In other words, a “fuzzy multiset”  $A$  should be characterized by a function  $X \rightarrow I \times \mathbb{N}$ . To distinguish these structures from fuzzy multisets, we will call them *multi-fuzzy* sets [16]. Given a multi-fuzzy set  $A$ , the expression  $A(x) = (i, n)$  “says” that there are  $n$  copies of  $x$  belong to  $A$  and with a degree equal to  $i$ .

Apart from their application to mathematics, multisets are really useful structures as interesting models of computations are built upon them. For instance, the chemical abstract machine of Berry and Boudol [4] is an abstract machine that is well-suited to model concurrent computation and manipulates *solutions*, which are finite multisets of *molecules* where a molecule is simply a term of an algebra.

Membrane computing is a model of computation that is built around the notion of multiset rewriting rules. More specifically, membrane computing is a computational paradigm that was inspired by the way cells live and function (see [12] for an overview of the field of membrane computing). Roughly speaking, a cell consists of a membrane that separates the cell from its environment. In addition, this

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<sup>3</sup>Multisets are also known as “bags,” “heaps,” “bunches,” “samples,” “occurrence sets,” “weighted sets,” and “firesets”—finitely repeated element sets.

membrane consists of compartments surrounded by porous membranes, which, in turn, may contain other compartments, and so on. At any moment, matter flows from one compartment to any neighboring one. In addition, the cell interacts with its environment in various ways (e.g., by dumping matter to its environment). Obviously, at any moment a number of processes occur in parallel (e.g., matter moves into a compartment, while energy is consumed in another compartment, etc.).

A P system is a conceptual computational device whose functionality is based on an abstraction of the cell. Thus, a P system consists of porous membranes that are populated with multisets of objects, which are usually materialized as strings of symbols. In addition, there are rules that are used to change the configuration of the system. A P system behaves more or less like a parser, which is clearly hard-wired to a particular grammar. Thus, a P system stops when no rule can be applied to the system. The result of the computation is always equal to the cardinality of the multiset that is contained in a designated compartment. Now, since rigid mathematical models employed in life sciences are not completely adequate for the interpretation of biological information, there have been various proposals to use fuzzy sets in the modeling of biological systems (e.g., see [1, 3]). Thus, it is quite reasonable to attempt the use of the theory of fuzzy sets in P systems. Indeed, such an attempt has been described in [16] that is generalized to a certain degree in this paper.

Structure of the paper. In what follows, I will define  $L$ -multi-fuzzy sets and  $L$ -fuzzy hybrid sets. Next, I will define the basic operations between such structures (e.g., union, sum, etc.). Also, I will give the definition of certain standard fuzzy-theoretic operators. By replacing multisets with either  $L$ -multi-fuzzy sets or  $L$ -hybrid sets in the definition of both P systems and the chemical abstract machine, we end up with fuzzy versions of these notational computing devices. We formally define these devices and briefly investigate their properties. The paper ends with the customary concluding remarks.

## 2. On $L$ -Multi Fuzzy Sets and $L$ -Fuzzy Hybrid Sets

One may say that multisets form an abstraction of the *token-type* distinction, which is the basis of the “token-token identity theory” [13], while (ordinary) sets are an abstraction of the denial of the token-type distinction. To make clear the essence of the token-type distinction, I will borrow an example from [13]. If one writes the word “dog” three times (i.e., “dog dog dog”), then she has written three instances, or tokens, of the one type of word. This observation necessitates a distinction between types (i.e., abstract general entities) and tokens (i.e., concrete particular objects and events). “A token of a type is a particular concrete exemplification of that abstract general type”. [13, page 59].

There is no question that three instances of the word “dog” are tokens of the “dog” type. However, there are many instances where one cannot make such a definitive statement regarding the type of some tokens. In particular, there are many cases where some token  $t$  is of type  $T$  to a certain degree. For example, consider the following glyphs:



Each of them depicts an “A,” however, each one is reminiscent of the “standard A” to some degree. For instance, the rightmost is less reminiscent of the “standard A”, while the leftmost glyph looks like an ordinary “A.” A text is clearly a multiset of letters, but if we are free to use any of these “As” to typeset a document, then we need a fuzzy multiset to *describe* the text. However, this is not a common practice and so we need a more restricted structure that better models real life situations.<sup>4</sup> As was noted above these structures will be called multi-fuzzy sets.

Clearly, it is too restrictive to demand that tokens are of some type to a degree, which is expressed by some number that belongs to the unit interval. More generally, we can assume that the likelihood degree is drawn from some frame  $L$ . Indeed, it is possible to define many other partially ordered sets that are frames. For example, Vickers [21] shows that finite observations on bit streams have the properties of a frame. Thus, the similarity degree would express the idea that an element resembles, in some way, an element of such a frame. Typically, a frame is defined as follows [21]:

**Definition 2.1.** A poset  $A$  is a *frame* iff

- i) every subset has a join
- ii) every finite subset has a meet
- iii) binary meets distribute over joins:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}.$$

Note that a frame is clearly a distributive lattice. So,  $L$ -multi-fuzzy sets are an extension of multi-fuzzy sets just like  $L$ -fuzzy sets [7] are an extension of fuzzy sets.

A.I. Kostrikin in his comments in the entry for the concept of duality in the *Encyclopaedia of Mathematics*<sup>5</sup> notes that “[d]uality is a very pervasive and important concept in (modern) mathematics.” One could argue that hybrid sets, and, therefore, fuzzy hybrid sets, extend multisets and their fuzzy counterparts to describe and/or to model dualities. Let us now proceed with the formal definition of  $L$ -fuzzy hybrid sets:

**Definition 2.2.** An  $L$ -fuzzy hybrid set  $\mathcal{A}$  is a mathematical structure that is characterized by a function  $\mathcal{A} : X \rightarrow L \times \mathbb{Z}$ , where  $L$  is a frame, and it is associated with a  $L$ -fuzzy set  $A : X \rightarrow L$ . More specifically, the equality  $\mathcal{A}(x) = (\ell, n)$  means that  $\mathcal{A}$  contains exactly  $n$  copies of  $x$ , where  $A(x) = \ell$ .

If we substitute  $\mathbb{Z}$  with  $\mathbb{N}$  in the previous definition, then the resulting structures will be called *L-multi-fuzzy sets*.

<sup>4</sup>Although our example makes it clear that fuzzy multisets are useful, still their usage example is not that realistic. Nevertheless, if we consider “identical” computational processes that may be similar to some *prototype* process to different degrees, then we have a situation where fuzzy multisets are useful (see [17] for more details regarding this idea).

<sup>5</sup>See <http://eom.springer.de/>.

Assuming that  $\mathcal{A}$  is an  $L$ -fuzzy hybrid set, then one can define the following two functions: the *multiplicity* function  $\mathcal{A}_m : X \rightarrow \mathbb{Z}$  and the *membership* function  $\mathcal{A}_\mu : X \rightarrow L$ . Clearly, if  $\mathcal{A}(x) = (\ell, n)$ , then  $\mathcal{A}_m(x) = n$  and  $\mathcal{A}_\mu(x) = \ell$ . Notice that it is equally easy to define the corresponding functions for an  $L$ -multi-fuzzy set.

I believe this is a good point to briefly express my prejudices and my intentions regarding the present work. Clearly, it is not my intention to develop an axiomatic set theory of  $L$ -fuzzy hybrid sets and  $L$ -multi-fuzzy sets, in the sense of the Zermelo-Fr ankel set theory, but rather a “naive” set theory in the sense that I will not present a precise axiomatization. Therefore, I plan to introduce only the basic set-theoretic operations and the basic properties of these sets. To begin with, let me now define the cardinality of an  $L$ -fuzzy hybrid set:

**Definition 2.3.** Assume that  $\mathcal{A}$  is an  $L$ -fuzzy hybrid set that draws elements from a universe  $X$ . Then its cardinality is defined as follows:

$$\text{card } \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes \mathcal{A}_m(x),$$

where  $\otimes : L \times \mathbb{Z} \rightarrow \mathbb{R}$  is a binary multiplication operator that is used to compute the product of  $\ell \in L$  times  $n \in \mathbb{Z}$ .

**Example 2.4.** If  $L = \mathbb{I} \times \mathbb{I}$  (i.e, when extending “intuitionistic” fuzzy sets, see [18]), then  $(i, j) \otimes n = in - jn$ .

**Remark 2.5.** When  $L$  is the unit interval, then  $\otimes$  is the usual multiplication operator.

The cardinality of a set is equal to the number of elements the set contains. Clearly, the previous definition is not in spirit with this assumption. However, hybrid sets may contain elements that occur a negative number of times. Thus, one may think that we should take this fact under consideration when computing the cardinality of a hybrid set and, more generally, the cardinality of an  $L$ -fuzzy hybrid set. So, it makes sense to introduce the notion of a *strong* cardinality defined as follows:

**Definition 2.6.** Assume that  $\mathcal{A}$  is an  $L$ -fuzzy hybrid set that draws elements from a universe  $X$ . Then its strong cardinality is defined as follows:

$$\text{card } \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes |\mathcal{A}_m(x)|,$$

where  $|\mathcal{A}_m(x)|$  denotes the absolute value of  $\mathcal{A}_m(x)$ .

For reasons of completeness I give below the definition of the cardinality of  $L$ -multi-fuzzy sets:

**Definition 2.7.** Assume that  $\mathcal{A}$  is an  $L$ -multi-fuzzy set that draws elements from a universe  $X$ . Then its cardinality is defined as follows:

$$\text{card } \mathcal{A} = \sum_{x \in X} \mathcal{A}_\mu(x) \otimes \mathcal{A}_m(x),$$

where  $\ell \otimes n$  is some binary operator that maps  $\ell \in L$  and  $n \in \mathbb{N}$  to some positive real number (since  $n \geq 0$ ).

In order to complete the presentation of the basic properties of fuzzy hybrid sets, it is necessary to define the notion of subsethood. Before, going on with this definition, I will introduce the (new) partial order  $\ll$  over  $\mathbb{Z}$ . In particular, if  $n, m \in \mathbb{Z}$ , then

$$\begin{aligned} n \ll m \quad \equiv \quad & (n = 0) \vee \\ & \left( (n > 0) \wedge (m > 0) \wedge (n \leq m) \right) \vee \\ & \left( (n < 0) \wedge (m > 0) \right) \vee \\ & (|n| \leq |m|). \end{aligned}$$

Note that here  $\wedge$  and  $\vee$  denote the classical logical conjunction and disjunction operators, respectively. In addition, the symbols  $\leq$  and  $<$  are the well-known ordering operators, and  $|n|$  is the absolute value of  $n$ .

**Example 2.8.** From the previous definition it should be obvious that  $0 \ll n$ , for all  $n \in \mathbb{Z}$ . Also,  $3 \ll 4$ ,  $-3 \ll 4$ , and  $-4 \ll -3$ .

But what kind of structure is the pair  $(\mathbb{Z}, \ll)$ ? The answer is easy with the help of the following result:

**Proposition 2.9.** *The relation  $\ll$  is a partial order.*

*Proof.* I have to prove that the relation  $\ll$  is reflexive, antisymmetric and transitive:

**Reflexivity:** Assume that  $a \in \mathbb{Z}$ . Then if  $a = 0$ ,  $a \ll a$  from the first part of the disjunction. If  $a < 0$ , then  $a \ll a$  from the fourth part of the disjunction and if  $a > 0$ , then  $a \ll a$  from the second part of the disjunction.

**Antisymmetry:** Assume that  $a, b \in \mathbb{Z}$ ,  $a \ll b$ , and  $b \ll a$ . Then if  $a = 0$  this implies that  $b = 0$  and so  $a = b$ . If  $a < 0$ , then it follows that  $b < 0$ ,  $|a| \leq |b|$ , and  $|b| \leq |a|$ , which implies that  $a = b$ . Similarly, if  $a > 0$ , then it follows that  $b > 0$ ,  $a \leq b$ , and  $b \leq a$ , which implies that  $a = b$ .

**Transitivity:** Assume that  $a, b, c \in \mathbb{Z}$ ,  $a \ll b$ , and  $b \ll c$ . Then if  $a = 0$ , then clearly  $a \ll c$ . If  $a < 0$  and  $b < 0$ , then either  $c < 0$  or  $c > 0$ , but since  $|b| \leq |c|$ , this implies that  $a \ll c$ . If  $a > 0$  and  $b > 0$ , then  $c > 0$  and since  $b \leq c$  this implies that  $a \ll c$ . If  $a < 0$  and  $b > 0$ , then since  $b \ll c$ , this implies that  $c > 0$ , which means that  $a \ll c$ .  $\square$

Note that  $a \gg b$  is an alternative form of  $b \ll a$ , which will be used in the rest of this paper. Let us now proceed with the definition of the notion of subsethood for  $L$ -fuzzy hybrid sets:

**Definition 2.10.** Assume that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{Z}$  are two  $L$ -fuzzy hybrid sets. Then  $\mathcal{A} \subseteq \mathcal{B}$  if and only if  $\mathcal{A}_\mu(x) \subseteq \mathcal{B}_\mu(x)$  and  $\mathcal{A}_m(x) \ll \mathcal{B}_m(x)$  for all  $x \in X$ .

Remark that for all  $\ell_1, \ell_2 \in L$ ,  $\ell_1 \sqsubseteq \ell_2$  if  $\ell_1$  is “less than or equal” to  $\ell_2$  in the sense of the partial order defined over  $L$ . The definition of subsethood for  $L$ -multi-fuzzy sets is more straightforward:

**Definition 2.11.** Assume that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}$  are two  $L$ -multi-fuzzy sets. Then  $\mathcal{A} \subseteq \mathcal{B}$  if and only if  $\mathcal{A}_\mu(x) \sqsubseteq \mathcal{B}_\mu(x)$  and  $\mathcal{A}_m(x) \leq \mathcal{B}_m(x)$  for all  $x \in X$ .

### 3. Basic Set Operations

The basic operations between sets are their union and their intersection. A third operation, viz. set sum, is meaningful only for multisets. Also, since both  $L$ -multi-fuzzy sets and  $L$ -fuzzy hybrid sets are actually generalizations of fuzzy sets, one should be able to define the  $\alpha$ -cuts of such sets. I will start by defining the basic set operations between  $L$ -multi-fuzzy sets.

**3.1. Set Operations of  $L$ -Multi-Fuzzy Sets.** Let me first present the definitions of union and intersection of  $L$ -multi-fuzzy sets:

**Definition 3.1.** Assuming that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}$  are two  $L$ -multi-fuzzy sets, then their union, denoted  $\mathcal{A} \cup \mathcal{B}$ , is defined as follows:

$$(\mathcal{A} \cup \mathcal{B})(x) = \left( \mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \max\{\mathcal{A}_m(x), \mathcal{B}_m(x)\} \right),$$

where  $a \sqcup b$  is the join of  $a, b \in L$ .

**Definition 3.2.** Assuming that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}$  are two  $L$ -multi-fuzzy sets, then their intersection, denoted  $\mathcal{A} \cap \mathcal{B}$ , is defined as follows:

$$(\mathcal{A} \cap \mathcal{B})(x) = \left( \mathcal{A}_\mu(x) \sqcap \mathcal{B}_\mu(x), \min\{\mathcal{A}_m(x), \mathcal{B}_m(x)\} \right),$$

where  $a \sqcap b$  is the meet of  $a, b \in L$ .

I will now define the sum of two  $L$ -multi-fuzzy sets:

**Definition 3.3.** Suppose that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{N}$  are two  $L$ -multi-fuzzy sets. Then their sum, denoted  $\mathcal{A} \uplus \mathcal{B}$ , is defined as follows:

$$(\mathcal{A} \uplus \mathcal{B})(x) = \left( \mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \mathcal{A}_m(x) + \mathcal{B}_m(x) \right).$$

Although it is crystal clear, it is necessary to say that  $\sqcup$  and  $\sqcap$  are operators that are part of the definition of the frame  $L$ . And as such they have a number of properties (e.g., they are idempotent, etc., see [21, p. 15] for details) that, naturally, affect the properties of the operations defined so far. Indeed, these operations have the following properties:

**Theorem 3.4.** For any three  $L$ -multi-fuzzy sets  $\mathcal{A}, \mathcal{B}, \mathcal{C} : X \rightarrow L \times \mathbb{N}$  the following equalities hold:

i) *Commutativity:*

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} &= \mathcal{B} \cup \mathcal{A} \\ \mathcal{A} \cap \mathcal{B} &= \mathcal{B} \cap \mathcal{A} \\ \mathcal{A} \uplus \mathcal{B} &= \mathcal{B} \uplus \mathcal{A};\end{aligned}$$

ii) *Associativity:*

$$\begin{aligned}\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} \\ \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} \\ \mathcal{A} \uplus (\mathcal{B} \uplus \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \uplus \mathcal{C};\end{aligned}$$

iii) *Idempotency:*

$$\begin{aligned}\mathcal{A} \cup \mathcal{A} &= \mathcal{A} \\ \mathcal{A} \cap \mathcal{A} &= \mathcal{A};\end{aligned}$$

iv) *Distributivity:*

$$\begin{aligned}\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}) \\ \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C});\end{aligned}$$

v) *Distributivity of sum:*

$$\begin{aligned}\mathcal{A} \uplus (\mathcal{B} \cup \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \cup (\mathcal{A} \uplus \mathcal{C}) \\ \mathcal{A} \uplus (\mathcal{B} \cap \mathcal{C}) &= (\mathcal{A} \uplus \mathcal{B}) \cap (\mathcal{A} \uplus \mathcal{C});\end{aligned}$$

*Proof.* i) Although this is easy, I will prove all cases:

$$\begin{aligned}(\mathcal{A} \cup \mathcal{B})(z) &= \left( \mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z), \max\{\mathcal{A}_m(z), \mathcal{B}_m(z)\} \right) \\ &= \left( \mathcal{B}_\mu(z) \sqcup \mathcal{A}_\mu(z), \max\{\mathcal{B}_m(z), \mathcal{A}_m(z)\} \right) \\ &= (\mathcal{B} \cup \mathcal{A})(z)\end{aligned}$$

$$\begin{aligned}(\mathcal{A} \cap \mathcal{B})(z) &= \left( \mathcal{A}_\mu(z) \sqcap \mathcal{B}_\mu(z), \min\{\mathcal{A}_m(z), \mathcal{B}_m(z)\} \right) \\ &= \left( \mathcal{B}_\mu(z) \sqcap \mathcal{A}_\mu(z), \min\{\mathcal{B}_m(z), \mathcal{A}_m(z)\} \right) \\ &= (\mathcal{B} \cap \mathcal{A})(z)\end{aligned}$$

$$\begin{aligned}(\mathcal{A} \uplus \mathcal{B})(z) &= \left( \mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z), \mathcal{A}_m(z) + \mathcal{B}_m(z) \right) \\ &= \left( \mathcal{B}_\mu(z) \sqcup \mathcal{A}_\mu(z), \mathcal{B}_m(z) + \mathcal{A}_m(z) \right) \\ &= (\mathcal{B} \uplus \mathcal{A})(z)\end{aligned}$$



ii) I will prove only the first case as the others can be proved similarly:

$$\begin{aligned} (\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}))(z) &= \left( \mathcal{A}_\mu(z) \sqcup \left( \mathcal{B}_\mu(z) \sqcup \mathcal{C}_\mu(z) \right), \max \left\{ \mathcal{A}_m(z), \max \left\{ \mathcal{B}_m(z), \mathcal{C}_m(z) \right\} \right\} \right) \\ &= \left( \left( \mathcal{A}_\mu(z) \sqcup \mathcal{B}_\mu(z) \right) \sqcup \mathcal{C}_\mu(z), \max \left\{ \max \left\{ \mathcal{A}_m(z), \mathcal{B}_m(z) \right\}, \mathcal{C}_m(z) \right\} \right) \\ &= ((\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C})(z) \end{aligned}$$

iii) As in the previous case, I will prove only the first case as the other can be proved similarly:

$$\begin{aligned} (\mathcal{A} \cup \mathcal{A})(z) &= \left( \mathcal{A}_\mu(z) \sqcup \mathcal{A}_\mu(z), \max \left\{ \mathcal{A}_m(z), \mathcal{A}_m(z) \right\} \right) \\ &= \left( \mathcal{A}_\mu(z), \mathcal{A}_m(z) \right) \\ &= \mathcal{A}(z) \end{aligned}$$

iv) The proof of this case follows from the fact that the following equalities are true for any three elements of a frame:

$$\begin{aligned} x \sqcap (y \sqcup z) &= (x \sqcap y) \sqcup (x \sqcap z) \\ x \sqcup (y \sqcap z) &= (x \sqcup y) \sqcap (x \sqcup z) \end{aligned}$$

v) As with the previous case the proof for this case follows from the fact that for any  $x, y, z \in \mathbb{N}$  the following equalities hold:

$$\begin{aligned} x + \max\{y, z\} &= \max\{x + y, x + z\} \\ x + \min\{y, z\} &= \min\{x + y, x + z\} \end{aligned}$$

□

The  $\alpha$ -cut of a fuzzy subset is just a crisp set. Similarly, the  $\alpha$ -cut of an  $L$ -multi-fuzzy set has to be a multiset. Indeed, if  $[x]_n$  denotes a multiset that consists of only  $n$  copies of  $x$ , the following definition is in spirit with the general theory of fuzzy sets:

**Definition 3.5.** Suppose that  $\mathcal{A}$  is an  $L$ -multi-fuzzy set with universe set  $X$ , and that  $\alpha \in L$ . Then the  $\alpha$ -cut of  $\mathcal{A}$ , denoted by  ${}^\alpha\mathcal{A}$ , is the multiset

$${}^\alpha\mathcal{A} = \bigcup_{\substack{x \in X \\ \alpha \sqsubseteq \mathcal{A}_\mu(x)}} [x]_{\mathcal{A}_m(x)}.$$

Not so surprisingly, the properties of the  $\alpha$ -cut of  $L$ -multi-fuzzy sets are similar to those of plain fuzzy sets. These properties are summarized below:

**Theorem 3.6.** Assume that  $\mathcal{A}$  and  $\mathcal{B}$  are two  $L$ -multi-fuzzy sets with universe set  $X$ . Then the following properties hold:

- i) if  $\alpha \sqsubseteq \beta$ , then  ${}^\alpha\mathcal{A} \supseteq {}^\beta\mathcal{A}$  and
- ii)  ${}^\alpha(\mathcal{A} \cap \mathcal{B}) = {}^\alpha\mathcal{A} \cap {}^\alpha\mathcal{B}$ ,  ${}^\alpha(\mathcal{A} \cup \mathcal{B}) = {}^\alpha\mathcal{A} \cup {}^\alpha\mathcal{B}$ , and  ${}^\alpha(\mathcal{A} \uplus \mathcal{B}) = {}^\alpha\mathcal{A} \uplus {}^\alpha\mathcal{B}$ .

*Proof.* i) Let  $x \in X$  and  $\alpha \sqsubseteq \beta$ . If  $\mathcal{A}_\mu(x) \not\sqsupseteq \beta$ , then  ${}^\alpha\mathcal{A}(x) = {}^\beta\mathcal{A}(x)$ . If  $\alpha \sqsubseteq \mathcal{A}_\mu(x) \sqsubseteq \beta$ , then  ${}^\alpha\mathcal{A}(x) \geq {}^\beta\mathcal{A}(x)$ . If  $\alpha \not\sqsubseteq \mathcal{A}_\mu(x)$ , then  ${}^\alpha\mathcal{A}(x) = {}^\beta\mathcal{A}(x) = 0$ . Thus, for all possible cases  ${}^\alpha\mathcal{A}(x) \geq {}^\beta\mathcal{A}(x)$ , which means that  ${}^\alpha\mathcal{A} \supseteq {}^\beta\mathcal{A}$ .

ii) Assume that  ${}^\alpha(\mathcal{A} \cap \mathcal{B})(x) = n$ . Then this means that

$$\min\{\mathcal{A}_m(x), \mathcal{B}_m(x)\} = n.$$

Also, it implies that  $(\mathcal{A} \cap \mathcal{B})_\mu(x) \supseteq \alpha$  and hence  $\mathcal{A}_\mu(x) \cap \mathcal{B}_\mu(x) \supseteq \alpha$ . From this, one can immediately deduce that  $\mathcal{A}_\mu(x) \supseteq \alpha$  and  $\mathcal{B}_\mu(x) \supseteq \alpha$ . Suppose now that  $\mathcal{A}_m(x) = n_1$  and  $\mathcal{B}_m(x) = n_2$ . Then this means that  ${}^\alpha\mathcal{A}(x) = n_1$  and  ${}^\alpha\mathcal{B}(x) = n_2$  and so

$$\min\{{}^\alpha\mathcal{A}(x), {}^\alpha\mathcal{B}(x)\} = n.$$

□

**3.2. Set Operations of  $L$ -Fuzzy Hybrid Sets.** Loeb has shown that the set of all subsets of a given hybrid set with the subsethood relation do not form a lattice. This means that if  $f$  and  $g$  are two hybrid sets, then if they have lower bounds, they do not necessarily have a greatest lower bound. Similarly, if  $f$  and  $g$  have upper bounds, then they do not necessarily have a lowest upper bound. Practically, this means that given two hybrid sets  $f$  and  $g$ , one cannot define their union and their intersection. Fortunately, the sum of hybrid sets is a well-defined operation. Thus, we can easily extend this definition as follows:

**Definition 3.7.** Assume that  $\mathcal{A}, \mathcal{B} : X \rightarrow L \times \mathbb{Z}$  are two  $L$ -fuzzy hybrid sets. Then their sum, denoted  $\mathcal{A} \uplus \mathcal{B}$ , is defined as follows:

$$(\mathcal{A} \uplus \mathcal{B})(x) = \left( \mathcal{A}_\mu(x) \sqcup \mathcal{B}_\mu(x), \mathcal{A}_m(x) + \mathcal{B}_m(x) \right).$$

Let  $\{f_i\}$  denote a finite collection of hybrid sets with a common universe  $X$ , where each of these sets contains repeated occurrence of only one element  $x_i \in X$ . In addition, let us insist that no two  $f_i$  and  $f_j$  will have common elements. Also, let us denote with  $\uplus_i f_i$  the unique hybrid set that is the sum of all  $f_i$ . With these preliminary definitions, the road for the following definition has been paved:

**Definition 3.8.** Suppose that  $\mathcal{A}$  is an  $L$ -fuzzy hybrid set with universe  $X$  and that  $\alpha \in L$ . Then the  $\alpha$ -cut of  $\mathcal{A}$ , denoted by  ${}^\alpha\mathcal{A}$ , is the hybrid set  $\uplus_i f_i$ , where  $f_i(x_i) = \mathcal{A}_m(x)$  iff  $\alpha \sqsubseteq \mathcal{A}_\mu(x)$ , for all  $x_i \in X$ .

The  $\alpha$ -cut of  $L$ -fuzzy hybrid sets has the following properties:

**Theorem 3.9.** Assume that  $\mathcal{A}$  and  $\mathcal{B}$  are two  $L$ -fuzzy hybrid sets with universe the set  $X$ . Then the following properties hold:

- i) if  $\alpha \sqsubseteq \beta$ , then  ${}^\alpha\mathcal{A} \supseteq {}^\beta\mathcal{A}$
- ii)  ${}^\alpha(\mathcal{A} \uplus \mathcal{B}) = {}^\alpha\mathcal{A} \uplus {}^\alpha\mathcal{B}$ .

*Proof.* The proof is similar to the proof of Theorem 3.6 and is omitted. □

#### 4. General Fuzzy P Systems

In [16] the author has proposed fuzzified versions of P systems. The basic idea behind this particular attempt to fuzzify P systems is the substitution of one or

all ingredients of a P system with their fuzzy counterparts. From a purely computational point of view, it turns out that only P systems that process multi-fuzzy sets are interesting. The reason being the fact that these systems are capable of computing (positive) real numbers. By replacing the multi-fuzzy sets employed in the author's previous work with  $L$ -multi-fuzzy sets, the computational power of the resulting P systems will not be any "greater," nevertheless, these systems may be quite useful in modeling living organisms. But, things may get really interesting if we consider P systems with  $L$ -fuzzy hybrid sets, in general. Let us begin with the definition of these systems:

**Definition 4.1.** A general fuzzy P system is a construction

$$\Pi_{\text{FD}} = (O, \mu, w^{(1)}, \dots, w^{(m)}, R_1, \dots, R_m, i_0)$$

where:

- i)  $O$  is an alphabet (i.e., a set of distinct entities) whose elements are called *objects*;
- ii)  $\mu$  is the membrane structure of degree  $m \geq 1$ ; membranes are injectively labeled with succeeding natural numbers starting with one;
- iii)  $w^{(i)} : O \rightarrow L \times \mathbb{Z}$ ,  $1 \leq i \leq m$ , are  $L$ -fuzzy hybrid sets over  $O$  that are associated with each region  $i$ ;
- iv)  $R_i$ ,  $1 \leq i \leq m$ , are finite sets of multiset rewriting rules (called *evolution rules*) over  $O$ . An evolution rule is of the form  $u \rightarrow v$ ,  $u \in O^*$  and  $v \in O_{\text{TAR}}^*$ , where  $O_{\text{TAR}} = O \times \text{TAR}$ ,  $\text{TAR} = \{\text{here}, \text{out}\} \cup \{\text{in}_j | 1 \leq j \leq m\}$ . The effect of each rule is the removal of the elements of the left-hand side of each rule from the "current" compartment and the introduction of the elements of right-hand side to the designated compartments;
- v)  $i_0 \in \{1, 2, \dots, m\}$  is the label of an elementary membrane (i.e., a membrane that does not contain any other membrane), called the *output* membrane.

The really interesting thing with the systems described in [16] is that I haven't managed to find any limits on what can be actually computed. Remember, that a real number  $x \in \mathbb{R}$  is called *computable* if there is a computable sequence  $(r_n)_{n \in \mathbb{N}}$  of rational numbers which converges to  $x$  effectively, that is, for all  $n \in \mathbb{N}$ ,  $|x - r_n| < 2^{-n}$  (see [22, 24] for details). In other words, this means that not all real numbers are *computable*. However, one should not forget that the definition of *computability* is hard-wired to the computational capabilities of the Universal Turing Machine and the so called Church-Turing thesis, which dictates what can be and what cannot be computed. Now, the crucial question is whether there are any limits that prohibit the computation of certain numbers with fuzzy P systems? It seems that these system go beyond the Church-Turing barrier because their set of input values is drastically larger than that of the Turing machine. However, it is an open problem the determination of the exact computational power of these systems.

## 5. Conclusions

In this paper I have introduced  $L$ -multi-fuzzy sets and  $L$ -fuzzy hybrid sets as well as their basic operations. In addition, general fuzzy P systems have been

introduced, which can be used to compute real numbers. I do not believe that this is something really new—it is just another indication that the current theory of computation is simply inadequate to describe all computational phenomena. After all, this has been elegantly demonstrated by Stein in his thought provoking paper [14]. In addition, I believe that we need a paradigm shift in computer science so to encompass new “phenomena” and practices.

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