Mathematics of Multisets

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Abstract. This paper is an attempt to summarize the basic elements of the multiset theory. We begin by describing multisets and the operations between them, then we present hybrid sets and their operations. We continue with a categorical approach to multisets, and then we present fuzzy multisets and their operations. Finally, we present partially ordered multisets.

1 Introduction

Many fields of modern mathematics have been emerged by *violating* a basic principle of a given theory only because useful structures could be defined this way. For example, modern non-Euclidean geometries have been emerged by assuming that the *Parallel Axiom*[1] does not hold. Similarly, *multisets* [8,13,16] have been defined by assuming that for a given set A an element x occurs a finite number of times. Multisets are also known as "bags" (but many consider this term too vulgar...), "heap", "bunch", "sample", "occurrence set", "weighted set" and "fireset"—finitely repeated element set. An argument against the position that the term "bag" is too vulgar is that this term is a plain English word—something in which we put things to carry them around. Besides, in English language mathematical literature it is a tradition to use plain words –group, set, ring, ...– unlike other sciences, where people invent new long ones by sticking Greek and Latin words together. Also, we must note that the term "multiset" has been coined by N.G. de Bruijn [8]. The first person who actually used multisets is Richard Dedekind in his well-known paper "Was sind und was sollen die Zahlen?" ("The nature and meaning of numbers") [6]. This paper was published in 1888. The reader interested to read a rather complete account of the development of multiset theory should read Blizard's excellent survey [5].

From a practical point of view multisets are very useful structures arising in many areas of mathematics and computer science. The prime factorization of an integer $n > 0$ is a multiset $\mathcal{N}$ whose elements are primes. Every monic polynomial $f(x)$ over the complex numbers corresponds in a natural way to the

* Dedicated to the fond memory of my brother Mikhail Syropoulos.

1 Which can be stated as follows: Given a point P not incident with line m, there is exactly one line incident with P and parallel to m.

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multiset $\mathcal{F}$ of its "roots". Other examples of multisets include the zeros and poles of meromorphic functions, the invariants of matrices in a canonical form, the invariants of finite Abelian groups, etc. The terminal strings of a context-free grammar form a multiset which is a set iff the grammar is unambiguous. Processes in an operating system can be thought of as multisets. The mathematical treatment of concurrency involves the use of multisets. In social sciences, multisets can be used to model social structures, etc.

There are three methods to define a set and we are recalling them now, since they will be heavily used in the rest of the text:

1. A set is defined by naming all its members (*the list method*). This method can be used only for finite sets. Set A, whose members are $a_1, a_2, \dots, a_n$, is usually written as
$$A = \{a_1, a_2, \dots, a_n\}.$$
2. A set is defined by a property satisfied by its members (*the rule method*). A common notation expressing this method is
$$A = \{x \mid P(x)\},$$
where the symbol $\mid$ denotes the phrase "such that", and $P(x)$ designates a proposition of the form "x has the property P."
3. A set is defined by a function, usually called the *characteristic function*, that declares which elements of a universal set X are members of set A and which are not. Set A is defined by its characteristic function, χ_A, as follows:
$$\chi_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

In what follows we present the definition of multisets and the basic operations between multisets. Moreover, we briefly present hybrid sets, i.e., multisets which may have negative integers as multiplicities as well as nonnegative integers. Then, we proceed with a categorical approach to multisets by defining categories of multisets. Next, we present fuzzy multisets and their operations. We finish by presenting pomsets and their basic operations.

2 Multisets and Their Operations

Ordinary sets are composed of pairwise different elements, i.e., no two elements are the same. If we relax this condition, i.e., if we allow multiple but finite occurrences of any element, we get a generalization of the notion of a set which is called a *multiset*.

There are two different kinds of sets with a finite number of repeated elements: sets with distinguishable repeated elements, e.g., people sharing a common property, and sets with indistinguishable repeated elements, e.g., a "soup" of elementary particles. Monro [12] calls the first kind of sets *multisets* and the second *multinumbers*. However, in order to avoid confusion, we will use the term multiset for Monro's multinumbers and the term *real* multisets for Monro's multisets.

Real multisets and multisets are associated with a (ordinary) set and an equivalence relation or a function, respectively. Here are the formal definitions:

Definition 1. *A real multiset $\mathcal{X}$ is a pair (X, ρ), where X is a set and ρ an equivalence relation on X. The set X is called the field of the real multiset. Elements of X in the same equivalence class will be said to be of the same sort; elements in different equivalence classes will be said to be of different sorts.*

Given two real multisets $\mathcal{X} = (X, \rho)$ and $\mathcal{Y} = (Y, \sigma)$, a morphism of real multisets is a function $f : X \to Y$ which respects sorts; that is, if $x, x' \in X$ and $x \,\rho\, x'$, then $f(x) \,\sigma\, f(x')$.

Definition 2. *Let D be a set. A multiset over D is just a pair $\langle D, f\rangle$, where D is a set and $f : D \to \mathbb{N}$ is a function.*

The previous definition is the characteristic function definition method for multisets.

Remark 1. Any ordinary set A is actually a multiset $\langle A, \chi_A\rangle$, where χ_A is its characteristic function.

Since multisets are sets with multiple but finite occurrences of any element, one can define a multiset by employing the list method. However, in order to avoid confusion we will use square brackets for multisets and braces for sets. In what follows we will employ the most suitable definition method for each case we encounter.

An important notion in set theory is the notion of a subset. Moreover, for ordinary sets there are certain operations one can perform between sets, such as set intersection, union, etc. We proceed with the definitions of the notion of the subset of a multiset and the operations between multisets.

Definition 3. *Suppose that $\mathcal{A} = \langle A, f\rangle$ is a multiset; the subset B of A is called the support of $\mathcal{A}$ if for every x such that $f(x) > 0$ this implies that $x \in B$, and for every x such that $f(x) = 0$ this implies that $x \notin B$.*

It is clear that the characteristic function of B can be specified as:

$$\chi_B(x) = \min(f(x), 1).$$

Example 1. Given the multiset $\mathcal{A} = [a, a, a, b, c, c]$, then its support is the set $\{a, b, c\}$.

Definition 4. *Assume that $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$ are two multisets. We say that $\mathcal{A}$ is a sub-multiset of $\mathcal{B}$, denoted $\mathcal{A} \subseteq \mathcal{B}$ if for all $a \in A$ we have*

$$f(a) \leq g(a).$$

$\mathcal{A}$ is called a proper sub-multiset of $\mathcal{B}$, denoted $\mathcal{A} \subset \mathcal{B}$, if in addition for some $a \in A$ we have

$$f(a) < g(a).$$

Obviously, it follows that for any two multisets, $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Definition 5. *Let $\mathcal{A} = \langle A, f \rangle$ be a multiset; $\mathcal{A}$ is the empty multiset if for all $a \in A$, $f(x) = 0$.*

Definition 6. *Suppose that $\mathcal{A} = \langle A, f \rangle$ is a multiset; its cardinality, denoted* card$(\mathcal{A})$*, is defined as*

$$\mathrm{card}(\mathcal{A}) = \sum_{a \in A} f(a).$$

If A is a set, then $\mathcal{P}^A$ is the set of all multisets which have A as their support set. Moreover, A is the smallest non-empty multiset in $\mathcal{P}^A$ in the sense that if $\mathcal{B} \in \mathcal{P}^A$, then

$$\mathrm{card}(\mathcal{B}) \geq \mathrm{card}(A).$$

We are now turning our attention to the operations between multisets. We define, in this order, the sum, the removal, the union and the intersection of two multisets.

Definition 7. *Suppose that $\mathcal{A} = \langle A, f \rangle$ and $\mathcal{B} = \langle A, g \rangle$ are two multisets. Their sum, denoted $\mathcal{A} \uplus \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h \rangle$, where for all $a \in A$:*

$$h(a) = f(a) + g(a).$$

It can be easily shown that the multiset sum operation has the following properties:

1. Commutative: $\mathcal{A} \uplus \mathcal{B} = \mathcal{B} \uplus \mathcal{B}$;
2. Associative: $(\mathcal{A} \uplus \mathcal{B}) \uplus \mathcal{C} = \mathcal{A} \uplus (\mathcal{B} \uplus \mathcal{C})$;
3. There exists a multiset, the null multiset $\emptyset$, such that $\mathcal{A} \uplus \emptyset = \mathcal{A}$.

It is important to note that there exists no inverse and multiset sum is not idepotent.

Definition 8. *Suppose that $\mathcal{A} = \langle A, f \rangle$ and $\mathcal{B} = \langle A, g \rangle$ are two multisets. The removal of multiset $\mathcal{B}$ from $\mathcal{A}$, denoted $\mathcal{A} \ominus \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h \rangle$, where for all $a \in A$:*

$$h(a) = \max\Big(f(a) - g(a), 0\Big).$$

Definition 9. *Suppose that $\mathcal{A} = \langle A, f \rangle$ and $\mathcal{B} = \langle A, g \rangle$ are two multisets. Their union, denoted $\mathcal{A} \cup \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h \rangle$, where for all $a \in A$:*

$$h(a) = \max\Big(f(a), g(a)\Big).$$

Definition 10. *Suppose that* $\mathcal{A} = \langle A, f\rangle$ *and* $\mathcal{B} = \langle A, g\rangle$ *are two multisets. Their intersection, denoted* $\mathcal{A} \cap \mathcal{B}$*, is the multiset* $\mathcal{C} = \langle A, h\rangle$*, where for all* $a \in A$*:*

$$h(a) = \min\Big(f(a), g(a)\Big).$$

The following properties can be easily established for union, intersection, and sum of multisets:

1. Commutativity:
$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$$
$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A};$$

2. Associativity:
$$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$$
$$\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C};$$

3. Idempotency:
$$\mathcal{A} \cup \mathcal{A} = \mathcal{A}$$
$$\mathcal{A} \cap \mathcal{A} = \mathcal{A};$$

4. Distributivity:
$$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$$
$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C});$$

5.
$$\mathcal{A} \uplus (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \uplus \mathcal{B}) \cup (\mathcal{A} \uplus \mathcal{C})$$
$$\mathcal{A} \uplus (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \uplus \mathcal{B}) \cap (\mathcal{A} \uplus \mathcal{C});$$

6.
$$\mathcal{A} \cap (\mathcal{A} \uplus \mathcal{B}) = \mathcal{A}$$
$$\mathcal{A} \cup (\mathcal{A} \uplus \mathcal{B}) = \mathcal{A} \uplus \mathcal{B};$$

7.
$$\mathcal{A} \uplus \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) \uplus (\mathcal{A} \cap \mathcal{B}).$$

Let $\mathcal{A} = \langle X, f\rangle$ be a multiset and $B \subseteq X$. We are interested in forming the multiset $\mathcal{C} = \langle X, g\rangle$, where

$$g(x) = \begin{cases} f(x), & \text{if } x \in B, \\ 0, & \text{if } x \notin B, \end{cases}$$

or, in other words, $g(x) = f(x) \cdot \chi_B(x)$.

A closely related problem is that of forming a multiset by removing all the elements from $\mathcal{A}$ which are in the set B. That is, we are interested in forming the multiset $\mathcal{D} = \langle X, h\rangle$, where

$$h(x) = \begin{cases} 0, & \text{if } x \in B, \\ f(x), & \text{if } x \notin B, \end{cases}$$

which can be expressed compactly as follows:

$$h(x) = f(x) \cdot (1 - \chi_B(x)), \forall x \in B.$$

However, one may note that $1 - \chi_B(x)$ is the characteristic function of the *complement* set of B, denoted $\bar{B}$. So, the previous equation becomes

$$h(x) = f(x) \cdot \chi_{\bar{B}}(x), \forall x \in B.$$

Thus,

$$\mathcal{D} = \mathcal{A} \circledast \bar{B}.$$

We shall call the operation $\mathcal{A} \circledast B$ *multi-intersection*. In general, it holds that if $\mathcal{A} = \langle X, \chi_A \rangle$, $A \subseteq X$, and B is a set, then $\mathcal{A} \circledast B = A \cap B$. Moreover, the following properties hold:

$$\begin{aligned} A \circledast X &= A \\ A \circledast \emptyset &= \emptyset \\ (\mathcal{A}_1 \cap \mathcal{A}_2) \circledast B &= (\mathcal{A}_1 \circledast B) \cap (\mathcal{A}_2 \circledast B) \\ (\mathcal{A}_1 \cup \mathcal{A}_2) \circledast B &= (\mathcal{A}_1 \circledast B) \cup (\mathcal{A}_2 \circledast B) \end{aligned}$$

3 Hybrid Sets

Hybrid sets and new sets are generalization of multisets and sets respectively. In a hybrid set the multiplicity of an element can be either a negative number, zero, or a positive number. A new set is to hybrid sets what is a set to a multiset, i.e., a special case. We give now the definition of the hybrid set due to Loeb [9]:

Definition 11. *Given a universe U, any function $f : U \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, is called a hybrid set. The value of $f(u)$ is said to be the multiplicity of the element u. If $f(u) \neq 0$ we say that u is a member of f and we write $u \in f$; otherwise, we write $u \notin f$. We define the number of elements, $\#f$, to be the sum $\sum_{u \in U} f(u)$. f is said to be an $\#f$ (element) hybrid set.*

Hybrid sets are denoted by employing the list method and by inserting a bar to separate elements with negative multiplicity from those with a non-negative multiplicity. Elements occurring with a positive multiplicity are written on the left of the bar, and elements occurring with a negative multiplicity are written on the right of the bar.

Example 2. If $f = \{a, b, b \mid d, e, e\}$ is a hybrid set, then $f(a) = 1$, $f(b) = 2$, $f(d) = -1$, and $f(e) = -2$.

The empty hybrid set, denoted $\emptyset$, is the unique hybrid set for which all elements have multiplicity equal to zero. We can specify the empty hybrid set by $\{|\}$. The definition of subset-hood in the case of hybrid sets is also due to Loeb:

Definition 12. *Let f and g be hybrid sets. We say that f is a subset of g and that g contains f and we write $f \subseteq g$ if either $f(u) \ll g(u)$ for all $u \in U$, or $g(u) - f(u) \ll g(u)$ for all $u \in U$, where $\ll$ is a partial ordering of the integers defined as follows: $i \ll j$ iff $i \leq j$ and either $i < 0$ or $j \geq 0$.*

We proceed now with the definition of the various operations between hybrid sets.

Definition 13. *Assume that f and g are two hybrid sets over the same universe U. Then, their intersection, $f \cap g$, is the hybrid set h, such that $h(u) = \max(f(u), g(u))$, their union, $f \cup g$, is the hybrid set h, such that $h(u) = \min(f(u), g(u))$, and their sum, $f \uplus g$, is the hybrid set h, such that $h(u) = f(u) + g(u)$.*

We easily verify the correctness of these definitions by using the definition of subset-hood.

4 Categorical Models of Multisets

Let $\mathcal{C}$ be a category. A functor $E : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$. Thus, a presheaf on $\mathcal{C}$ is a contravariant functor. The presheaves on $\mathcal{C}$ with natural transformations as arrows form a category denoted $\mathbf{Psh}(\mathcal{C})$. Suppose that C is a set, i.e., a discrete category; then, the presheaf $F : C \to \mathbf{Set}$ denotes a multiset, since $F(c)$ is a set whose cardinality is equal to the number of times c occurs in the multiset. So, for any set C, the category $\mathbf{Psh}(C)$ denotes the category of all multisets of C. These remarks lead us to the definition of a category of all possible multisets:

Definition 14. *Category* **MSet** *is a category of all possible multisets.*

1. *The objects of the category consist of pairs (A, P), where A is a set and $P : A \to \mathbf{Set}$ a presheaf on A.*
2. *If (A, P) and (B, Q) are two objects of the category, an arrow between these objects is a pair (f, λ), where $f : A \to B$ is a function and $\lambda : P \to Q \circ f$ is a natural transformation, i.e., a family of functions.*
3. *Arrows compose as follows: suppose that $(A, P) \xrightarrow{(f,\lambda)} (B, Q)$ and $(B, Q) \xrightarrow{(g,\mu)} (C, R)$ are arrows of the category, then $(f, \lambda) \circ (g, \mu) = (g \circ f, \mu \times \lambda)$, where $g \circ f$ is the usual function composition and $\mu \times \lambda : P \to R \circ (g \circ f)$.*
4. *Given an object (A, P), the identity arrow is $(\text{id}_A, \text{id}_P)$.*

The last part of the definition is a kind of wreath product (see [4]). However, it is not clear at the moment how this definition fits into the general theory of wreath products.

This is not the only way one can categorically define multisets. Suppose that $F : A \to \mathbf{Set}$ is a presheaf and that A is a set. Then if we form the set $X = \bigcup_{i \in A} X_i$, where $X_i = F(i)$, we can define the function $p : X \to A$. This function is equivalent to the presheaf F. Moreover, $p^{-1}(a)$, i.e., the preimage of p, is the set of copies of a in the multiset. Now, we can define another category of all possible multisets:

Definition 15. *Category* **Bags** *is a category of all possible multisets.*

1. *The objects of the category consist of pairs (A, p), where $p : \bigcup_{i \in A} X_i \to A$.*

2. *An arrow between two objects* (A, p) *and* (B, q) *is a pair* (f, g)*, where* $f : A \to B$ *and* $g : X \to Y$*, such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[r]{} & B \end{array}$$

3. *Suppose that* $(A, p) \stackrel{(f,g)}{\longrightarrow} (B, q)$ *and* $(B, q) \stackrel{(f',g')}{\longrightarrow} (C, r)$ *are two arrows. Then* $(f, g) \circ (f', g') = (f' \circ f, g' \circ g)$ *such that in the following diagram*

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{g'} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ A & \xrightarrow[f]{} & B & \xrightarrow[f']{} & C \end{array}$$

the outer rectangle commutes iff the inner squares commute.
4. *Given an object* (A, p) *the identity arrow is* $(\mathrm{id}_A, \mathrm{id}_X)$*.*

It is obvious that categories **MSet** and **Bags** are *equivalent*. Moreover, one can study the properties of the categories **MSet** and **Bags**, but we feel this is not the appropriate place for such a presentation. We now investigate the way one can embed the category **Bags** into a Chu category [15].

Given an arbitrary object $\perp$ in a category $\mathbf{A}$, we construct the category $\mathbf{Chu}(\mathbf{A}, \perp)$ as follows:

1. The objects of $\mathbf{Chu}(\mathbf{A}, \perp)$ consist of triplets (A_1, r, A_2), where A_1, A_2 are objects in A and $r : A_1 \otimes A_2 \to \perp$ is an arrow in $\mathbf{A}$.
2. An arrow from (A_1, r, A_2) to (B_1, s, B_2) is a pair $(f, \hat{f})$, where $f : A_1 \to B_1$ and $\hat{f} : B_2 \to A_2$ are arrows in $\mathbf{A}$ such that the square

$$\begin{array}{ccc} A_1 \otimes B_2 & \xrightarrow{f \otimes \mathrm{id}_{B_2}} & B_1 \otimes B_2 \\ {\scriptstyle \mathrm{id}_{A_1} \otimes \hat{f}}\downarrow & & \downarrow{\scriptstyle s} \\ A_1 \otimes A_2 & \xrightarrow[r]{} & \perp \end{array}$$

commutes.
3. Arrow composition is defined pairwise.

If $\mathbf{A}$ is any $*$-autonomous category (see [1,2]), then $\mathbf{Chu}(\mathbf{A}, \perp)$ is $\mathbf{A^2}$, where $\perp$ is a dualizing object [3]. It is now easy to define a full embedding of **Bags** into $\mathbf{Chu}(\mathbf{Rel}, 1)$, where **Rel** is the category of sets and binary relations between them and 1 is any singleton set.

We start by defining the object part of the functor:

Definition 16. ***(Object part)*** *Functor $\mathcal{M}$ maps each object (A, p) of* **Bags** *into the Chu space $(A, \tilde{p}, X)$, where $X = \mathrm{dom} p$, and $\tilde{p}$ is the relation obtained from function p and a p X_i iff the multiplicity of a is equal to the cardinality of X_i.*

We now proceed with the arrow part of the functor:

Definition 17. ***(Arrow part)*** *Let (A, p) and (B, q) be two objects of the category* **Bags**. *Moreover, suppose that $(A, p) \xrightarrow{(f,g)} (B, q)$ is an arrow between these objects; then $\mathcal{M}(f, g) = (\tilde{f}, \tilde{g}^{-1})$, where $\tilde{f}$ is the relation obtained from function f and $\tilde{g}^{-1}$ the inverse of the relation obtained from function g.*

The possible consequences of this embedding are explored in [15].

5 Fuzzy Multisets

Fuzzy set theory has been introduced as a means to deal with vagueness in mathematics. The theory is well-established and we will not get into the trouble of presenting it. We just note that fuzzy set theory was an attempt to develop a formal apparatus to involve a partial membership in a set, mainly to arm people in the modeling of empirical objects and facts. In other words, fuzzy set theory is, sort to say, a generalization of the notion of set membership.

Definition 18. *Suppose that X is a set. Any function $A : X \to \mathrm{I}$, where $\mathrm{I} = [0, 1]$, is called a fuzzy subset of X. Function A is usually called the membership function of the fuzzy subset A.*

Fuzzy multisets have been introduced by Yager [16] and have been studied by Miyamoto [10,11] and others. A fuzzy multiset of some set X is just a multiset of $X \times \mathrm{I}$. We are now defining summation of fuzzy multisets:

Definition 19. *If $\mathcal{A} = \langle X \times \mathrm{I}, f\rangle$ and $\mathcal{B} = \langle X \times \mathrm{I}, g\rangle$ are two fuzzy multisets, then their sum, denoted $\mathcal{A} \uplus \mathcal{B}$, is the fuzzy multiset $\mathcal{C} = \langle X \times \mathrm{I}, h\rangle$, where for all $(x, \mu_x) \in X \times \mathrm{I}$:*

$$h(x, \mu_x) = f(x, \mu_x) + g(x, \mu_x).$$

As in the case of *crisp*[2] multisets, there is more that one way to define a fuzzy multiset. In order to define the basic operations between fuzzy multisets, we define fuzzy multisets by the list method. If $\mathcal{A} = \{(x_i, \mu_i)\}_{i=1,\ldots,p}$ be a fuzzy multiset, then we can write the same set as $\mathcal{A} = \{\{\mu_{11}, \ldots, \mu_{1\ell_1}\}/x_1, \ldots, \{\mu_n, \ldots, \mu_{n\ell_n}\}/x_n\}$. Note that $\{\mu_{11}, \ldots, \mu_{1\ell_1}\}$ is actually a multiset of I. Next, we rearrange the multisets $\{\mu_{11}, \ldots, \mu_{1\ell_1}\}$ so that the elements appear in decreasing order. Finally, we need to add zeroes so that the length of all multisets $\{\mu_{11}, \ldots, \mu_{1\ell_1}\}$ is the same. This representation is called the *graded sequence*. To make things clear we give an example:

[2] In fuzzy set theory the term crisp is used to characterize anything that is non-fuzzy.

Example 3. Let

$$\mathcal{A} = [(a, 0.2), (b, 0.5), (b, 0.1), (a, 0.2), (a, 0.3), (d, 0.7)]$$

be a fuzzy multiset. Then its graded sequence is

$$\mathcal{A} = [\{0.3, 0.2, 0.2\}/a, \{0.5, 0.1, 0\}/b, \{0.7, 0, 0\}/d]$$

In case we want to perform certain operations on two or more fuzzy multisets, all multisets $\{\mu_{11}, \ldots, \mu_{1\ell_1}\}$ must have the same length. Moreover, even if one fuzzy multiset does not contain an element c we must add an entry of the form $\{\underbrace{0, 0, \ldots, 0}_{p \text{ times}}/c\}$, where p is the length of all other multisets. We are now giving the definitions of the various operations between fuzzy multisets:

Definition 20. *Assume that* $\mathcal{A} = [[\mu_{1p}, \ldots, \mu_{11}]/x_1, \ldots, [\mu_{np}, \ldots, \mu_{n1}]/x_n]$ *and* $\mathcal{B} = [[\mu'_{1p}, \ldots, \mu'_{11}]/x_1, \ldots, [\mu'_{np}, \ldots, \mu'_{n1}]/x_n]$ *are two fuzzy multisets; then*

1. $\mathcal{A} \subseteq \mathcal{B}$ *iff for every* x_i, $\mu_{ij} \le \mu'_{ij}$, $j = 1, \ldots, p$.
2. $\mathcal{A} = \mathcal{B}$ *iff for every* x_i, $\mu_{ij} = \mu'_{ij}$, $j = 1, \ldots, p$.
3. $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, *where* $\mathcal{C} = [[\mu''_{1p}, \ldots, \mu''_{11}]/x_1, \ldots, [\mu''_{np}, \ldots, \mu''_{n1}]/x_n]$ *iff for every* x_i, $\mu''_{ij} = \max(\mu'_{ij}, \mu_{ij})$, $j = 1, \ldots, p$.
4. $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$, *where* $\mathcal{C} = [[\mu''_{1p}, \ldots, \mu''_{11}]/x_1, \ldots, [\mu''_{np}, \ldots, \mu''_{n1}]/x_n]$ *iff for every* x_i, $\mu''_{ij} = \min(\mu'_{ij}, \mu_{ij})$, $j = 1, \ldots, p$.

When the functions max and min are replaced by a t-norm t and a t-conorm s respectively, we obtain the definitions for $\cap_t$ and $\cup_s$, respectively. The union and intersection of arbitrary fuzzy multisets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy the following laws:

1. Commutativity.
$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$$
$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$$
2. Associativity.
$$(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$$
$$(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$$
3. Distributivity.
$$(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$$
$$(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$$

Next, we define the α-cut for fuzzy multisets. We first recall the notion of the α-cut for a fuzzy set:

Definition 21. *Let* U *be a set, let* C *be a partially ordered set and let* $A : U \to C$. *For* $\alpha \in C$, *the* α*-cut of* A, *is* $A^{-1}(\uparrow \alpha) = \{u \in U \mid A(u) \ge \alpha\}$. *The subset of* U *will be denoted by* A_α.

Definition 22. *Assume that* $\mathcal{A} = \langle X \times \mathrm{I}, f\rangle$ *is a fuzzy multiset, and that* $\alpha \in (0, 1]$. *Then* $\mathcal{A}_\alpha = \langle X, f'\rangle$, *i.e., the* α*-cut of* $\mathcal{A}$, *is a multiset such that*

$$f'(x) = \sum_{\mu_x \ge \alpha} f(x, \mu_x).$$

Consequently, the α-cut of a fuzzy multiset is just a multiset.

Given a fuzzy multiset $\mathcal{A} = \langle X \times \mathrm{I}, h\rangle$ and function $f : X \to Y$ we can define two images:

$$f[\mathcal{A}] = \uplus_{x\in\mathcal{A}}\{f(x)\}$$
$$f(\mathcal{A}) = \bigcup_{x\in\mathcal{A}}\{f(x)\}$$

Note that in case the fuzzy multiset is just a fuzzy subset, the second image corresponds to the *extension principle* of fuzzy set theory.

6 Partially Ordered Multisets

Partially ordered multisets (or just pomsets) have been used by Pratt [14] as a means to model concurrency. In this model a process is a set of pomsets. Here we will only present the definition of a pomset and the basic operations between pomsets. The reader interested in learning more on their use on modeling concurrency is refereed to Pratt's paper. The following definition of pomset is due to Gischer [7] and is copied verbatim from Pratt's paper:

Definition 23. *A labeled partial order (lpo) is a 4-tuple* $(V, \Sigma, \leq, \mu)$ *consisting of*

1. *a vertex set* V*, typically modeling events;*
2. *an alphabet* Σ *(for symbol set), typically modeling actions such as the arrival of integer 3 at port Q;*
3. *a partial order* $\leq$ *on* V*, with* $e \leq f$ *typically being interpreted as event* e *necessarily preceding event* f *in time; and*
4. *a labeling function* $\mu : V \to \Sigma$ *assigning symbols to vertices, each labeled event representing an occurrence of the action labeling it, with the same action possibly having multiply occurrences, that is,* μ *need not be injective.*

A pomset is then the isomorphism class of an lpo, denoted $[V, \Sigma, \leq, \mu]$.

Now we are ready to define the basic operations between pomsets:

Definition 24. *Assume that* $p = [V, \Sigma, \leq, \mu]$ *and* $p' = [V', \Sigma', \leq', \mu']$ *are two pomsets. Then:*

1. *their concurrence* $p||p'$ *is the pomset* $[V \cup V', \Sigma \cup \Sigma', \leq \cup \leq', \mu \cup \mu']$*, where* V *and* V' *are assumed to be disjoint;*
2. *their concatenation* $p;p'$ *is as for concurrence except that instead of* $\leq \cup \leq'$ *the partial order is taken to be* $\leq \cup \leq' \cup (V \times V')$*; and*
3. *their orthocurrence* $p \times p'$ *is the pomset* $[V \times V', \Sigma \times \Sigma', \leq \times \leq', \mu \times \mu']$.

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