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FUZZY SETS AND FUZZY RELATIONAL STRUCTURES AS CHU SPACES

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Chu spaces, which derive from the Chu construct of $*$-autonomous categories, can be used to represent most mathematical structures. Moreover, the logic of Chu spaces is linear logic. Most efforts to incorporate fuzzy set theory into the realm of linear logic are based on the assumption that fuzzy and linear negation are identical operations. We propose an incorporation based on the opposite assumption and we provide an interpretation of some linear connectives. Furthermore, we show that it is possible to represent any fuzzy relational structure as a Chu space by means of the functor G.

Keywords: Chu spaces, Fuzzy sets and Fuzzy relational structures

1. Introduction

Linear logic[1] is a logic which has been invented by Jean-Yves Girard while he was working on the properties of coherent spaces. Linear logic is not just another logic, it is an improvement of classical logic, i.e., linear logic attempts and solves most of the problems of classical logic. From its discovery linear logic has found many application in computer science, e.g., it is employed in the description of concurrent systems, in the design of new programming languages that don't need garbage collection, proof theory, etc.

Chu spaces[2] are relatively new objects of mathematics which derive from the Chu construct of $*$-autonomous categories[3] ([4] is a comprehensive recent account of the Chu construct). Chu spaces have been successfully used in many fields of science:

- in the modeling of constraint-interval fuzzy set as a means of fuzzy decision making,[5]
- in the mathematical description of information flow,[6]
- in the description of concurrency,[7]
- in physics,[8]
- in philosophy,[9]
- in mathematics as a uniform representation of the objects of mathematical practice[10] and of any relational structure,[11] etc.

It is interesting to note that the logic of Chu spaces is linear logic.[12]

There have been some attempts to incorporate fuzzy logic into linear logic.[13,14] All attempts are based on the assumption that linear negation and fuzzy negation should be treated as identical operations. Our approach departs from this assumption—we propose an incorporation of fuzzy logic into linear logic via Chu spaces by distinguishing fuzzy and linear negation. This distinction is justified by the fact that linear negation is a derived operation, while fuzzy negation is a basic operation.

In this paper we provide an elementary introduction to the notions and concepts associated with linear logic, then we propose a representation of fuzzy subsets of some universal set as Chu spaces and we provide an representation of the connectives of intuitionistic linear logic. Moreover, we provide an representation of fuzzy negation as an operation between Chu spaces. Next, we provide a construction by means of it one can represent any fuzzy relational structure as a Chu space.

2. What is Linear Logic?

Linear logic is not just another logic, it is an improvement of classical logic. The basic problem of classical logic, that is being solved by linear logic, is that of *static propositions*—in classical logic a proposition is either true or false, but this can never change. In general, this holds for all mathematical propositions, e.g., consider the proposition $3 > 2$. However, there are propositions whose truth is *temporal*, e.g., the sentence *it's Christmas today* is true only once, or in general every 365 days or so.

But this isn't the only problem of classical logic. Consider the following statements:

$D \triangleq$ one dollar
$M \triangleq$ a pack of Marlboros
$C \triangleq$ a pack of Camels

Moreover, suppose that the statements $D \Rightarrow M$ and $D \Rightarrow C$ mean that we can buy a pack of Marlboros with one dollar and that we can buy a pack of Camel's with one dollar, respectively. Then in classical logic we are allowed to conclude that

$$D \Rightarrow (M \wedge C)$$

That is, with one dollar we can buy a pack of Marlboros and a pack of Camel's! Of course this paradox is due to the interpretation of the connective $\wedge$ ("and"): if it means a *choice* between the two it is obviously true, but if it means simultaneous possession of both packs, then it is obviously false. These two interpretations are implemented in linear logic as two different conjunction connectives: $\&$ and $\otimes$ respectively. Moreover, these connectives have their *duals*: $\oplus$ (dual of $\&$) expresses the choice between two possible types of action and $\parr$ (dual of $\otimes$) expresses a dependency between two types of action.

Speaking of actions, linear logic assumes that a proposition is either a *situation* or an *action*/*reaction*. Situations correspond to eternal truths, while actions and reactions to propositions whose truth is temporal.[*] In linear logic we are not allowed to use any proposition more than once or to ignore it. However, this rule does not apply to situations, if we use them in a controlled way. The new connectives (or *modalities*) $!$ and $?$ permit us to consume and to produce, respectively, as many copies of some situation as we like. Furthermore, linear logic provides the special *null-ary* connectives (or *constants*) $\mathbf{1}$, $\bot$, $\top$, and $\mathbf{0}$ which are the *neutral propositions* of the connectives $\otimes$, $\parr$, $\&$, and $\oplus$ respectively. Linear negation is a defined operation, while linear implication can be defined in terms of the connective $\parr$ as: $A \multimap B = A^{\perp} \parr B$. Note that it is possible to define the connectives $\otimes$ and $\parr$ in terms of $\multimap$ and linear negation.

The following example, taken from,[16] illustrates the use of the linear connectives and their meaning:

Example 2.1 *Suppose for a fixed \$5 price a restaurant will provide a hamburger, a Coke, as many French fries as you like, onion soup or salad (your choice), and pie or ice cream (some else's choice). One may encode this information in the linear logic formula beside the menu:*

Fixed – Price Menu: \$5	$(D \otimes D \otimes D \otimes D \otimes D)$
Hamburger	$\multimap$
Coke	$[H \otimes C \otimes !F \otimes (O \,\&\, S) \otimes (O \oplus I)]$
All the french fries	
you can eat	
Onion Soup or Salad	
Pie or Ice Cream	
(depending on availability)	

♢

3. Fuzzy Subsets as Chu Spaces

Let Σ be an arbitrary set without structure (or better: we aren't interested in the structure it may have). A Σ-Chu space is just a triplet $(X, r, \mathfrak{A})$, where X and $\mathfrak{A}$ are arbitrary sets and $r : X \times \mathfrak{A} \to \Sigma$. Function r relates the elements of X with the elements of $\mathfrak{A}$. For example, if $\mathfrak{A}$ stands for the set of open subsets of X, then for $x \in X$ and $A \in \mathfrak{A}$ the expression $r(x, A)$ is equal to one if x belongs to the open subset A, and zero otherwise. This way it is possible to represent topological spaces and other relational structures in general.

[*]The reader should not confuse linear logic with any temporal logic, as the first deals with the notion of time in a rather deep way.[15]

Remark 3.1 Categorically speaking, a Σ-Chu space is just an object of the category $\mathbf{Chu}(\Sigma)$, i.e., the category $\mathbf{Chu}(\mathbf{Set}, \Sigma)$ where **Set** is the category of sets and functions, and Σ is a set called the *dualizing object*.

Consider the I-Chu space $(X, r, \mathfrak{A})$, where $\mathrm{I} = [0, 1]$. Suppose that X is some universal set and $\mathfrak{A}$ a set of fuzzy subsets of the universal set, then $r(x, A)$, where $x \in X$ and $A \in \mathfrak{A}$, denotes the *degree* to which x is a *member* of the fuzzy subset A. We call such a Chu space a *fuzzy Chu space* (or *FCS*, for short). Note that in general the elements of $\mathfrak{A}$ are not necessary elements of the set $\mathcal{F}(X) = I^X$, i.e., $\mathfrak{A}$ is an arbitrary set, which by means of a bijection, corresponds to some set $V \subseteq \mathcal{F}(X)$. Since, fuzzy subsets usually denote *properties*, it is useful to think of r as a table that relates the various *individuals* with their *properties*. The following example makes clear this idea.

Example 3.2 Let the set $X = \{x_1, x_2, \ldots, x_m\}$ denote the contestants of a beauty contest and the set $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ denote the *qualities* each contestant may have. Moreover, suppose that the following table provides an estimate of the degree to which each contestant has each quality.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \ldots & A_n \end{array} \\ \begin{pmatrix} r_{1,1} & r_{1,2} & \ldots & r_{1,n} \\ r_{2,1} & r_{2,2} & \ldots & r_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{m,1} & r_{m,2} & \ldots & r_{m,n} \end{pmatrix} \end{array}$$

Then $r(x_i, A_j) = r_{i,j}$, and so our beauty contest can be described by the Chu space $(X, r, \mathfrak{A})$. $\diamondsuit$

Suppose that $\mathcal{A} = (X, r, \mathfrak{A})$ is a FCS and that the function $\check{r} : \mathfrak{A} \to I^X$ is an injection, then the FCS $(X, \mathfrak{A})$ with $r(x, A)$ being defined implicitly as $A(x)$, provides an alternative, more natural, representation of a set and (some of) its fuzzy subsets.

The dual of the FCS $\mathcal{A} = (X, r, \mathfrak{A})$, denoted as $\mathcal{A}^{\perp}$, is defined to be the FCS $(\mathfrak{A}, r^{\smile}, X)$, where $r^{\smile}(A, x) = r(x, A)$. So, the dual space of a FCS is just another space which behaves identically but reverses the order of the sets. Someone could say that it is the mirror image of the original space. The dual of a Chu space corresponds to linear negation.

Suppose that $\mathcal{A} = (X, r, \mathfrak{A})$ is a FCS. Moreover, suppose that the set $\bar{\mathfrak{A}}$ consists of the *complements* of the elements of $\mathfrak{A}$, then a bijection $\varphi : \mathfrak{A} \to \bar{\mathfrak{A}}$, defines the FCS $\bar{\mathcal{A}} = (X, \bar{r}, \bar{\mathfrak{A}})$, such that $r(x, A) = 1 - \bar{r}(x, \varphi(A))$. The FCS $\bar{\mathcal{A}}$ is called the *complement* FCS. The complement of a FCS corresponds to fuzzy negation.

Let $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ be two FCS. Then a transformation from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functions $(f, \bar{f})$, such that $f : X \to Y$, $\bar{f} : \mathfrak{B} \to \mathfrak{A}$ and $s(f(x), B) = r(x, \bar{f}(B))$, for all $x \in X$ and $B \in \mathfrak{B}$. This condition is called the *adjointness condition*.

Proposition 3.3 *The adjointness condition in the case of FCS is equivalent to the extension principle of fuzzy set theory.*

Proof. We first recall the *extension principle* of fuzzy set theory: Any given function $f : X \to Y$ induces two functions $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ and $f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$ such that $[f(A)](y) = \sup_{x|y=f(x)} A(x)$ for all $A \in \mathcal{F}(X)$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in \mathcal{F}(Y)$. The meaning of the adjointness condition is that if we have a function, $f : X \to Y$ then it induces the function $\bar{f} : \mathcal{F}(Y) \to \mathcal{F}(X)$ such $s(f(x), B) = r(x, \bar{f}(B))$. If we let $r(x, A) = A(x)$, then the adjointness condition becomes $[\bar{f}(B)](x) = B(f(x))$, which is exactly the extension principle. □

We have defined a faithful way to describe the fuzzy sets of a given universal set as Chu spaces. Moreover, it is now trivial to define the category $\mathbf{Chu}(I)$ as follows:

(i) The objects of the category are FCS.

(ii) The morphisms of the category are pairs of functions that fulfill the adjointness condition

(iii) Ajoint pairs $\mathcal{A} \xrightarrow{(f,\bar{f})} \mathcal{B} \xrightarrow{(g,\bar{g})} \mathcal{C}$, where $\mathcal{A} = (X, r, \mathfrak{A})$, $\mathcal{B} = (Y, s, \mathfrak{B})$, $\mathcal{C} = (Z, t, \mathfrak{C})$, compose via $(g, \bar{g}) \circ (f, \bar{f}) = (g \circ f, \bar{g} \circ \bar{f})$. This is obviously an adjoint pair since $t((g \circ f)(x), C) = s(f(x), \bar{g}(C)) = r(x, (\bar{g} \circ \bar{f})(C))$.

(iv) The identity morphisms are pairs of identity functions.

Following[2] we provide a representation of the various linear connectives.

Definition 3.4 Suppose that $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$ are two FCSs, then the FCS

$$\mathcal{A} \multimap \mathcal{B} = (\mathcal{B}^{\mathcal{A}}, t, X \times \mathfrak{B})$$

where $t(f, (x, B)) = s(f(x), B)$, is the space of all transformations from $\mathcal{A}$ to $\mathcal{B}$. In order to define the other linear connectives we use the fact that we can define both $\otimes$ (tensor product) and ⅋ (par) in terms of $\multimap$ and linear negation. Here are the details:

- $\mathcal{A} ⅋ \mathcal{B} = \mathcal{A}^{\perp} \multimap \mathcal{B} = (\mathcal{B}^{\mathcal{A}^{\perp}}, \tau, \mathfrak{A} \times \mathfrak{B})$, where $\tau(f, (A, B)) = s(f(A), B)$ and $f : \mathfrak{A} \to Y$, and
- $\mathcal{A} \otimes \mathcal{B} = (\mathcal{A} \multimap \mathcal{B}^{\perp})^{\perp} = (X \times Y, \breve{t}, (\mathcal{B}^{\perp})^{\mathcal{A}})$, where $\breve{t}((x, y), f) = \breve{s}(f(x), y)$ and $f : X \to \mathfrak{B}$.

Next, we define the *sum* and the *product* of any two FCS. Since, their definition make use of the concept of the *direct sum*, or just sum, of two sets A and B, denoted as $A + B$; we take this opportunity to remind the reader that $A + B = \{0\} \times A \cup \{1\} \times B$.

Definition 3.5 The sum of two FCS $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, denoted as $\mathcal{A} \oplus \mathcal{B}$, is the triplet $(X + Y, t, \mathfrak{A} \times \mathfrak{B})$, where $t((x, y), (0, A)) = r(x, A)$ and $t((x, y), (1, B)) = s(y, B)$.

Definition 3.6 The product of two FCS $\mathcal{A} = (X, r, \mathfrak{A})$ and $\mathcal{B} = (Y, s, \mathfrak{B})$, denoted as $\mathcal{A}\ \&\ \mathcal{B}$, is the triplet $(X \times Y, t', \mathfrak{A} + \mathfrak{B})$, where $t((0, x), (A, B)) = r(x, A)$ and $t((1, y), (A, B)) = s(y, B)$.

Moreover, the modality ! (pronounced: *ofcourse*) can be defined in the following way:

Definition 3.7 Let A be an arbitrary set and let $\mathrm{MF}(A)$ be

$$\{M : A \to \mathbb{N}; M(a) > 0 \text{ for finitely many } a \in A\}$$

Let $\mathcal{A} = (X, r, \mathfrak{A})$ be a Chu space. We write $\mathcal{M}(\mathcal{A})$ for the set of pairs $(\bar{\varphi}, \hat{\varphi})$ with $\bar{\varphi} : X \to [0, 1]$ and $\hat{\varphi} : X \to \mathrm{MF}(\mathfrak{A})$ satisfying

$$r(x, A) = \bar{\varphi}(x)$$

for all $x \in X$ and $A \in \hat{\varphi}(x)$. Then the Chu space $!\mathcal{A}$ is defined as follows:

$$!\mathcal{A} = (X, r_!, \mathcal{M}(\mathcal{A})).$$

Moreover, $r_!(x, \varphi) = \bar{\varphi}(x)$.

We define now the four null-ary operators. $\bot = (I, \pi_1, 1)$, where $1 = \{0\}$ and $\pi_1(i, 0) = i$ for all $i \in I$, and $\mathbf{1} = \bot^\bot$ $\mathbf{0} = (\emptyset, !, 1)$ and $\top = \mathbf{0}^\bot$.

We have provided a representation of all connectives of intuitionistic linear logic as Chu spaces. Since, the logic of Chu spaces is the intuitionistic linear logic (and by this we mean that the proof of a linear formula can be represented by Chu spaces), it is now possible to exploit the possibility to employ fuzzy sets in linear logic reasoning.

4. Fuzzy Relational Structures and Chu Spaces

A *fuzzy relational structure* is a pair (A, ρ), where A is any (crisp) set and $\rho \in I^{A^n}$ ($n \in \mathbb{N}$). For example, a binary fuzzy relation S over some set B, i.e., $S : B \times B \to I$, is denoted by (B, S). If (X, R) and (Y, Q) are fuzzy relational structures of order n, and $h : X \to Y$ a function, then we say that h is a *relation morphism* iff $R(x_1, \ldots, x_n) \leq Q(h(x_1), \ldots, h(x_n))$ (see[17] for more details).

Proposition 4.1 *For any two fuzzy relational structures (A, ρ) and (B, σ) a relation morphism $f : A \to B$ induces a function $f : \mathcal{F}(A^n) \to \mathcal{F}(B^n)$, such that $f(\rho)(\mathbf{b}) \leq \sigma(\mathbf{b})$, where $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.*

Proof. Function $f : A \to B$ induces a function $f : A^n \to B^n$ in the obvious way. Moreover, this new function induces a function $f : \mathcal{F}(A^n) \to \mathcal{F}(B^n)$ (obviously, $\rho \in \mathcal{F}(A^n)$ and $\sigma \in \mathcal{F}(B^n)$). Now according to the extension principle of fuzzy set theory, we get that $f(\rho)(\mathbf{b}) = \sup_{\mathbf{a}|\mathbf{b}=f(\mathbf{a})} \rho(\mathbf{a})$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. We demand that a morphism between two fuzzy relational structures is a function so that

$$f(\rho)(\mathbf{b}) \leq \sigma(\mathbf{b})$$

which is equivalent to

$$\begin{aligned} \sup_{\mathbf{a}|\mathbf{b}=f(\mathbf{a})} \rho(\mathbf{a}) &\leq \sigma(\mathbf{b}) \Leftrightarrow \\ \rho(\mathbf{a}) &\leq \sigma(f(\mathbf{a})) \end{aligned}$$

□

Corollary 4.2 *For any two relation morphisms* $(A,\rho) \xrightarrow{f} (B,\sigma) \xrightarrow{g} (C,\tau)$*, it holds that* $(g \circ f)(\rho)(\mathbf{c}) \leq \tau(\mathbf{c})$.

Proof. The morphisms $(A,\rho) \xrightarrow{f} (B,\sigma) \xrightarrow{g} (C,\tau)$, imply that $f(\rho)(\mathbf{b}) \leq \sigma(\mathbf{b})$ and $g(\sigma)(\mathbf{c}) \leq \tau(\mathbf{c})$. Based one these facts, the proof is easy:

$$\begin{aligned}
(g \circ f)(\rho)(\mathbf{c}) &= g\Big(f(\rho)\Big)(\mathbf{c}) \\
&= \sup_{\mathbf{b}|\mathbf{c}=g(\mathbf{b})} f(\rho)(\mathbf{b}) \\
&\leq \sup_{\mathbf{b}|\mathbf{c}=g(\mathbf{b})} \sigma(\mathbf{b}) \\
&= g(\sigma)(\mathbf{b}) \\
&\leq \tau(\mathbf{c})
\end{aligned}$$

□

It is now possible to define a category, $\mathbf{FStr}_n$, of fuzzy relational structures of order n as follows:

(i) The objects of the category will be all the pairs (A,ρ), where A is a (crisp) set and $\rho \in I^{A^n}$.

(ii) For any two objects (A,ρ) and (B,σ) a relation morphism is an arrow between them.

(iii) Arrow composition is the composition of relation morphisms.

(iv) The identity morphism for a relational structure (A,ρ), $\rho \in I^{A^n}$, is a function id_ρ such that $\rho(\mathbf{a}) = \rho(\mathrm{id}_\rho(\mathbf{a}))$.

Pratt has provided a construction, more specifically a functor $F : \mathbf{Str}_n \to \mathbf{Chu}(2^n)$, by which one can transform any (crisp) relational structure into an object of the category $\mathbf{Chu}(2^n)$ (see[11] for more details). We extend the results of Pratt by defining the functor $G : \mathbf{FStr}_n \to \mathbf{Chu}(I^n)$. The object part of it is defined as follows:

Definition 4.3 If (A,ρ) is a fuzzy relational structure, then the functor G sends it to the triplet (A,r,X), where X is a set that consists of n-tuples of fuzzy subsets of A, such that for all $x \in X$, $\prod_i x_i \subseteq \bar{\rho}$. Moreover, let $r : A \times X \to I^n$ be defined as follows: $r(a,x)_i = x_i(a)$. (This definition is an immediate result of the *cylindrical extension principle*.)

The following is an immediate result of the definition.

Corollary 4.4 *G is injective on objects.*

We now must find what is the relation between the original fuzzy relational structure and the new Chu space.

Proposition 4.5 *Suppose that* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$*, then for all* $x \in X$*:*

$$\rho(\mathbf{a}) \leq \max\Big\{\bar{r}(a_i,x)_i, i = 1,\ldots,n\Big\}.$$

Proof.

$$\begin{aligned}\rho(\mathbf{a}) &= 1-\bar{\rho}(\mathbf{a}) \\ &\le 1-\min\Big\{x_1(a_1),\ldots,x_n(a_n)\Big\} \\ &= \max\Big\{1-x_1(a_1),\ldots,1-x_n(a_n)\Big\} \\ &= \max\Big\{1-r(a_1,x)_1,\ldots,1-r(a_n,x)_n\Big\} \\ &= \max\Big\{\bar{r}(a_i,x)_i, i=1,\ldots,n\Big\}.\end{aligned}$$

□

Note, that this means that the definition of Pratt in[11] is a special case of our definition.

We define now the arrow part of the functor.

Definition 4.6 Suppose that $f : A \to B$ is a function which induces the function $f^{-1} : \mathcal{F}(B) \to \mathcal{F}(A)$, such that $f^{-1}(B_i)(a) = B_i(f(a))$. Now, we demand that the functor G maps any function $f : A \to B$ to the pair of functions (f, f^{-1}). Obviously, this means that the pair of functions must fulfill the transformation condition between any two FCS, i.e.,

$$s(f(a), y) = r(a, f^{-1}(y)),$$

which holds.

We prove the following important facts:

Theorem 4.7 *The functor G is faithful and full.*

Proof. That the functor is full is a direct consequence of Corollary 4.4. That it is faithful is a direct consequence of its definition: for any $f_1, f_2 : (A, \rho) \to (B, \sigma)$, it follows that if $G(f_1) = G(f_2)$, then $(f_1, f_1^{-1}) = (f_2, f_2^{-1})$ and so $f_1 = f_2$. □

We conclude with an example of a fuzzy relational structure and the way one can represent it as an object of some category $\mathbf{FStr}_n$ and so as a Chu space.

Example 4.8 Because fuzzy logic is applied to a wide range of fields, not everyone agrees on the form the basic logic connectives should have. For example one may opt to represent logical disjunction as the $\max$ function or as the product of the two fuzzy logical values, etc. Suppose that X is a crisp set then the sextuple $(\mathcal{F}(X), \vee, \wedge, \neg, 0, 1)$ represents a DeMorgan algebra, where $\vee$ corresponds to disjunction, $\wedge$ to conjunction, $\neg$ to negation and 0 and 1 to the bottom and top elements of the algebra respectively. Each such sextuple can be represented as by (I, ρ), where ρ is a set of quadruplets. The collection of all such algebras forms a category $\mathbf{FStr}_4$. Consequently, it is possible to map every fuzzy DeMorgan algebra to some Chu space. ♢

5. Conclusions

We have showed that it is possible to represent fuzzy sets of some universal set as Chu spaces. Moreover, we have described a mechanism by means of it one can represent any fuzzy relational structure as a Chu space. The important question that has to be answered is: "What are the benefits of this approach?" The obvious answer to this question is that we get an immediate way to embed linear logic in virtually all areas of science. Consequently, the operators of linear logic get a concrete meaning for different cases. Since, Chu categories are large categories they provide a uniform framework to deal with different objects of mathematics that become objects of the same category. Another important aspect of this representation is that we can get a new insight in areas where Chu spaces have already been applied successfully, e.g., concurrency, quantum mechanics, etc.

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