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Categorical Models of Multisets

Apostolos SYROPOULOS

Greek Molecular Computing Group
366, 28th October Str.
GR-671 00 Xanthi, Greece

E-mail: apostolo@obelix.ee.duth.gr

Abstract. The mathematical representation of multisets as objects of Chu categories is a very promising development in the theory of multisets; it provides the framework to model computational and non-computational "phenomena" in a very natural way. In this paper we show how we can represent multisets as Chu Spaces and also we give some interesting examples.

1. Introduction

In many scientific disciplines we encounter groups of objects (e.g., people, elementary particles, etc.), that share a common property. In most cases we are only interested in the number of objects that have this particular property. For example, consider a group of people. Then if we are interested in their smoking habits we can say that n out of m people are smokers and, naturally, $m - n$ are non-smokers. Since such cases occur very often, we need a (formal) mathematical structure to model this kind of data. Such a structure exists and in general it is called a *multiset* (see [13] for an overview of the theory of multisets). Multisets form a generalization of sets: "identical" elements can occur a finite number of times. If we consider a multiset of people and a multiset of elementary particles, we observe that there are two forms of multisets: one where repeated elements are distinguishable and one where repeated elements are indistinguishable. We call the former multisets and the latter pure multisets.

Multisets have found many applications in computer science; for example, they are used in database theory [7], they are used to provide a semantical description of some form of the π-calculus [5], they are used in membrane computing [10] (an exciting new theory of computation), etc. In addition, a collection of papers on applications of multisets in theoretical computer science can be found in [4].

Category theory has been heavily used in computer science mainly because it offers an abstract way to view things. Chu categories are a particularly interesting case, because they have found so many application in computer science. For example, they have been used as models of concurrency [12], as models of fuzzy sets [9], as models of information flow [2], as a model of linear logic [6], etc. Consequently, it is interesting to see whether it is possible to model multisets as objects of some Chu category.

Structure of the paper. In this paper we begin with a formal definition of both multisets and pure multisets. We continue with the definition of categories of all possible multisets and pure multisets, respectively. Next, we define functors by which we map multisets and pure multisets to objects of (different) Chu categories. Moreover, we briefly describe the properties of these functors. We conclude with a description of possible applications of multisets (pure or not) as objects of Chu categories.

2. Mathematical Representations of Multisets

In this section we formally define multisets and pure multisets. We start with multisets (the definitions are from [8]):

Definition 2.1. A multiset $\mathcal{X}$ is a pair (X, ρ), where X is a set and ρ an equivalence relation on X. The set X is called the field of the multiset. Elements of X in the same equivalence class will be said to be of the same sort; elements in different equivalence classes will be said to be of different sorts.

Remark 1. The pair $(X, \emptyset)$, where $\emptyset$ denotes the empty relation and X is a set, is actually an ordinary set.

Given two multisets $\mathcal{X} = (X, \rho)$ and $\mathcal{Y} = (Y, \sigma)$, a morphism of multisets is a function $f : X \to Y$ that respects sorts; that is, if $x, x' \in X$ and $x \;\rho\; x'$, then $f(x) \;\sigma\; f(x')$. Most obviously, multisets and multiset morphisms form a category which we call **Mul**. Let us now proceed with pure multisets.

Definition 2.2. Let D be a set. A pure multiset over D is just a pair $\langle D, f\rangle$, where D is a set and $f : D \to \mathbb{N}$ is a function.

Remark 2. Any ordinary set A is actually the pure multiset $\langle A, \chi_A\rangle$, where χ_A is the characteristic function of A.

Let $\mathcal{C}$ be a category. A functor $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$. Thus, a presheaf on $\mathcal{C}$ is a contravariant functor. The presheaves on $\mathcal{C}$ with natural transformations as arrows form a category denoted $\mathbf{Psh}(\mathcal{C})$. Suppose that C is a set (i.e., a discrete category); then the presheaf $F : C \to \mathbf{Set}$ denotes a pure multiset, since $F(c)$ is a set whose cardinality is equal to the number of times c occurs in the pure multiset. So, for any set C, the category $\mathbf{Psh}(C)$ denotes the category of all pure multisets of C.

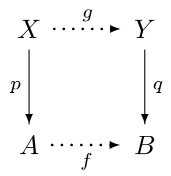
Suppose that $F : A \to \mathbf{Set}$ is a presheaf and that A is a set. Then if we form the set $X = \bigcup_{i\in A} X_i$, where $X_i = F(i)$, we can define the function $p : X \to A$. This function is equivalent to the presheaf F. Moreover, $p^{-1}(a)$ (i.e., the preimage of p)

is the set of copies of a in the pure multiset. Now, we can define a category of all possible pure multisets:

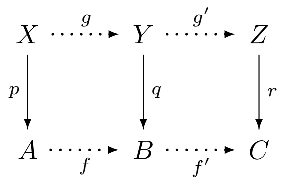
Definition 2.3. Category **Bags** is a category of all possible pure multisets.

i) The objects of the category are pairs (A, p), where $p : \bigcup_{i \in A} X_i \to A$.

ii) An arrow between two objects (A, p) and (B, q) is a pair (f, g), where $f : A \to B$ and $g : X \to Y$, such that the following diagram commutes:



iii) Suppose that $(A, p) \stackrel{(f,g)}{\longrightarrow} (B, q)$ and $(B, q) \stackrel{(f',g')}{\longrightarrow} (C, r)$ are two arrows; then $(f, g) \circ (f', g') = (f' \circ f, g' \circ g)$ such that in the following diagram



the outer rectangle commutes if and only if the inner squares commute.

iv) The arrow $(A, p) \stackrel{(\mathrm{id}_X, \mathrm{id}_A)}{\longrightarrow} (A, p)$ is the identity arrow.

We have provided categorical representations of both forms of multisets. We will now focus on how we can represent multisets as objects in some Chu category.

3. Multisets as Objects of Chu Categories

The Chu construct is a way to create a $*$-autonomous category from an autonomous category. The term autonomous category is an alternative term for a symmetrical monoidal closed category, while a $*$-autonomous category is an autonomous category with a given duality functor. The Chu construction is described in [1] and it is named after P.-H. Chu, a student of Michael Barr. Let us now give the definition of the Chu construct:

Definition 3.1. Given an arbitrary object $\perp$ in a category **A**, we construct the category $\mathbf{Chu}(\mathbf{A}, \perp)$ as follows:

i) The objects of $\mathbf{Chu}(\mathbf{A}, \perp)$ consist of triplets (A_1, r, A_2), where A_1, A_2 are objects in A and $r : A_1 \otimes A_2 \to \perp$ is an arrow in **A**.

ii) An arrow from (A_1, r, A_2) to (B_1, s, B_2) is a pair $(f, \hat{f})$, where $f : A_1 \to B_1$ and $\hat{f} : B_2 \to A_2$ are arrows in $\mathbf{A}$ such that the square

$$\begin{array}{ccc} A_1 \otimes B_2 & \xrightarrow{f\otimes \mathrm{id}_{B_2}} & B_1 \otimes B_2 \\ {\scriptstyle \mathrm{id}_{A_1}\otimes\hat{f}}\downarrow & & \downarrow{\scriptstyle s} \\ A_1 \otimes A_2 & \xrightarrow[r]{} & \bot \end{array} \tag{1}$$

commutes.

iii) Arrow composition is defined pairwise.

iv) For any object (A_1, r, A_2), the identity morphism is the pair $(\mathrm{id}_{A_1}, \mathrm{id}_{A_2})$, where id_{A_1} and id_{A_2} are the identity morphisms of A_1 and A_2 in $\mathbf{A}$.

We have found that pure multisets can be represented as objects in $\mathbf{Chu}(\mathbf{Rel}, 1)$, where **Rel** is the category of sets and binary relations between them and 1 a singleton set. Moreover, pure multisets can be represented as objects in $\mathbf{Chu}(\mathbf{Set}, 2^2)$, where **Set** is the category of sets and functions between them. Note that one can omit the category if no confusion arises (e.g., one can write $\mathbf{Chu}(K)$ instead of $\mathbf{Chu}(\mathbf{A}, K)$).

The objects in $\mathbf{Chu}(\mathbf{Rel}, 1)$ are triplets (X, R, A), where X and A are arbitrary sets and R is a binary relation in $X \times A$, i.e., $R \subseteq X \times A$. If $x \in X$ is *related* to some $a \in A$, we denote this by $x\ R\ a$. Otherwise, we simply write $x \not R\ a$.

Suppose that $\mathcal{A} = (X, R, A)$ and $\mathcal{B} = (Y, S, B)$ are two objects in $\mathbf{Chu}(\mathbf{Rel}, 1)$; then a transformation between them is just a pair of relations (f, g), where $f \subseteq X \times Y$ and $g \subseteq B \times A$, such that

$$\begin{array}{ccc} X \otimes B & \xrightarrow{f\otimes \mathrm{id}_{B}} & Y \otimes B \\ {\scriptstyle \mathrm{id}_{X}\otimes g}\downarrow & & \downarrow{\scriptstyle S} \\ X \otimes A & \xrightarrow[R]{} & 1 \end{array} \tag{2}$$

We mention now the following important result:

Proposition 3.1. *The following diagram*

$$\begin{array}{ccc} X & \xrightarrow{R} & A \\ f\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{S} & B \end{array} \tag{3}$$

is identical to diagram 2.

Proof. Since functor $\otimes$ in **Rel** is just the Cartesian product of sets (i.e., the categorical product in **Set**), so $a \otimes b \to \imath$ is just $a \to b$. $\square$

The above proposition can be stated more generally as follows:

Proposition 3.2. *If* $\mathbf{A}$ *is any* $*$*-autonomous category, then* $\mathbf{Chu}(\mathbf{A}, \perp)$ *is* $\mathbf{A^2}$, *where* $\perp$ *is a dualizing object.*

From Proposition 3.1 we conclude that diagram 3 is equivalent to the following equation:

$$(S \circ f)(x) = (g \circ R)(x), \forall x \in X. \tag{4}$$

Since we want to represent multisets as objects in $\mathbf{Chu}(\mathbf{Rel}, 1)$, we need to define a functor $\mathcal{M}$. We start by defining the object part of the functor:

Definition 3.2. (**Object part**) Functor $\mathcal{M}$ maps each object (A, p) in **Bags** into the Chu space $(A, \tilde{p}, X)$, where $X = \operatorname{dom} p$, and $\tilde{p}$ is the graph of the function p, and $a\ p\ X_i$ if and only if the multiplicity of a is equal to the cardinality of X_i.

We now proceed with the arrow part of the functor:

Definition 3.3. (**Arrow part**) Let (A, p) and (B, q) be two objects in **Bags**. Moreover, suppose that $(A, p) \stackrel{(f,g)}{\longrightarrow} (B, q)$ is an arrow between these objects. Then $\mathcal{M}(f, g) = (\tilde{f}, \tilde{g}^{-1})$, where $\tilde{f}$ is the graph of the function f and $\tilde{g}^{-1}$ the inverse of the graph of the function g.

The properties of the functor are summarized in the following assertion:

Theorem 3.1. *Functor* $\mathcal{M}$ *is injective on objects and faithful.*

Proof. Let (f, g) and (f', g') be two parallel morphisms in **Bags** such that $(f, g) \neq (f', g')$; then $\mathcal{M}(f, g) \neq \mathcal{M}(f', g')$, since the graphs of two different functions are different. This proves that $\mathcal{M}$ is faithful. That $\mathcal{M}$ is injective on objects is obvious from its definition. □

It is easy to verify that **Mul** is actually a sub-category of the category $\mathbf{Str}_2$ (i.e., the category of all binary relational structures). In general, the category of category of all n-ary relational structures (denoted $\mathbf{Str}_n$) is defined as follows:

Definition 3.4. For any ordinal n, an n-ary relational structure (X, ρ) consists of a set X, the carrier, and an n-ary relation $\rho \subseteq X^n$ on X. A homomorphism $f : (X, \rho) \to (Y, \sigma)$ between two such structures is a function $f : X \to Y$ between their underlying sets for which $f\rho \subseteq \sigma$. Here $f\rho$ denotes $\{f\mathbf{a} \mid \mathbf{a} \in \rho\}$, where $\mathbf{a}$ denotes $(a_0, \ldots, a_{n-1})$ and $f\mathbf{a}$ denotes $(fa_0, \ldots, fa_{n-1})$. We denote by $\mathbf{Str}_n$ the category formed by the n-ary relational structures and their homomorphisms.

In [11] it is shown that $\mathbf{Str}_n$ fully embeds into $\mathbf{Chu}(2^n)$; hence the objects in **Mul** can be represented as objects in $\mathbf{Chu}(2^2)$ using the same recipe. Here is the recipe: We start with a multiset (A, ρ), from this we construct the Chu space (A, u, R), where R consists of those pairs $r \in (2^A)^2$ of subsets of A for which $\prod_i r_i \in \hat{\rho}$. Note that $\hat{\rho} = 1 - \rho$. Now, let $u : A \times R \to 2^2$ satisfy $u(a, r)_i = 1$ if $a \in r_i$, and 0 otherwise. This completes our construction.

4. Discussion

Consider a group of people A. Moreover, suppose that we have asked this group of people to answer a particular question (e.g., "What is your favorite ice-cream flavor"?). Then the results of the survey can be represented by the multiset (A, ρ). If for any two individuals x and y it holds that $x \, \rho \, y$, then we are sure that both individuals have given the same response to the above question. In addition, if we have several such surveys, we can compose them using the following recipe: Construct the multiset that corresponds to each survey; create the corresponding Chu Space; by using the predefined operators (endo-functors) of the Chu category generate "composite" surveys. Let us now present two of these operators.

Assume that $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ are two Chu spaces. Then, $\mathcal{A} \oplus \mathcal{B} = (A + B, t, X \times Y)$ and $\mathcal{A} \,\&\, \mathcal{B} = (A \times B, t', X + Y)$. In the first case we have that

$$t(a, (x, y)) = r(a, x) \quad \text{and} \quad t(b, (x, y)) = s(b, x)$$

and in the second that

$$t'((a, b), x) = r(a, x) \quad \text{and} \quad t'((a, b), y) = s(b, x).$$

Let us see how we can interpret composite surveys generated with these two operators. Let $\mathcal{A}$ and $\mathcal{B}$ be two Chu spaces representing two surveys. Then $\mathcal{A} \oplus \mathcal{B}$ describes the combined survey where we can immediately tell what each individual answered to each survey. On the other hand, $\mathcal{A} \,\&\, \mathcal{B}$ is a combined survey form which we can tell which individuals answered a particular question.

All the processes that are active at any given moment in a computer system form a multiset. In addition, all processes that are spawned from a particular computer program belong to the same sort. Since we can represent any multiset as a Chu space, and Chu spaces have been proposed as models of concurrency, we can combine these facts to develop a more realistic model of concurrency that is based on Chu spaces and multisets. Similarly, we can "implement" existing models of concurrency by observing the same things. In what follows, we show how we can "implement" a particular model of concurrency using exactly these ideas.

The chemical abstract machine (or CHAM for short) [3], is a model of concurrency where processes are treated as molecules (in a chemical solution) that may interact. The basic laws that govern the behavior of the CHAM follow:

- *The Reaction Law.* An instance of the right-hand side of a rule can replace the corresponding instance of its left-hand side. Given a rule

  $$m_1, m_2, \ldots, m_k \longrightarrow m'_1, m'_2, \ldots, m'_l$$

  if $M_1, M_2, \ldots, M_k, M'_1, M'_2, \ldots, M'_l$ are instances of the m_i's and the m_j's by a common substitution, then

  $$\{\!|M_1, M_2, \ldots, M_k|\!\} \longrightarrow \{\!|M'_1, M'_2, \ldots, M'_l|\!\}$$

  where $\{\!|M_1, M_2, \ldots, M_k|\!\}$ denotes a solution.

- *The Chemical Law.* Reactions can be performed freely within any solution

$$\frac{S \to S'}{S \uplus S'' \to S' \uplus S''}$$

where $S \uplus S'$ is a new solution obtained from solutions S and S' by mixing-up them. Technically speaking, $S \uplus S'$ is a multiset with the property that if s occurs a times in S and b times in S', then it occurs $a+b$ times in $S \uplus S'$.

- *The Membrane Law.* A *membrane* ($\{\!|\cdot|\!\}$) is a mechanism by which we can transform a solution into a single molecule. This way we can allow sub-solutions evolve freely in any context

$$\frac{S \to S'}{\{\!|C[S]|\!\} \to \{\!|C[S']|\!\}},$$

where $C[S]$ denotes a *context*, i.e., a solution with a *hole* [ ], in which we can place another solution S. This operation is similar to the β-reduction of the λ-calculus.

- *The Airlock Law.* An airlock makes a membrane porus so that the solution contained in it can communicate with the outside world. The molecule constructor $\lhd$ builds a new molecule out of a molecule and a solution, so $m \lhd S$ is a molecule composed of the molecule m and the solution S. The following equation expresses this machinery:

$$\{\!|m|\!\} \uplus S \longleftrightarrow \{\!|m \lhd S|\!\}$$

Note that the double arrow means that we can go from left to right and vice versa.

Suppose that $(A, \tilde{p}, X)$ and $(B, \tilde{q}, Y)$ are two objects of $\mathbf{Chu}(\mathbf{Rel}, 1)$ representing to multisets. Then any arrow (f, g) between these two objects can be viewed as a reaction transforming the solution $(A, \tilde{p}, X)$ to $(B, \tilde{q}, Y)$. Mixing up two solutions in the sense of the *Chemical Law* is equivalent to the direct product (&) of two solutions. This simply means that we observe the same properties—nothing more, nothing less. Sub-solutions are actually subspaces of a particular space. It is not difficult to see how on can represent an ordinary set as a Chu space; therefore it is easy to derive the *Airlock Law*.

5. Conclusions

The derivation of an alternative mathematical representation of a well-known (mathematical) structure is not an easy task mainly because one has to justify the usefulness of this new representation. In this paper we described categorical models of multisets. In addition, we have shown how one can represent multisets as Chu spaces. Furthermore, we presented examples which demonstrated the usefulness of this new representation of multisets. We believe that our work can be proved useful to people trying to connect entities that seem to be unrelated, at least at first sight. For example, fuzzy multisets can be naturally represented if we generalize the Chu spaces that represent either multisets or fuzzy sets. Last, but not least, our ideas may have application to Membrane Computing as membranes may contain multisets of objects that interact.

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