VARIANTS OF INTUITIONISTIC FUZZY IMPLICATIONS*

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Fuzzy systems make heavy use of fuzzy logical operations to "deliver" their results. Among these operations, the most important one is the fuzzy implication, which is used to implement IF-THEN rules. Intuitionistic fuzzy sets are an extension of the notion of fuzzy sets where the complementary of the complementary of a fuzzy set is not the original fuzzy set. Here we present a number of intuitionistic fuzzy implication operators that are based on ordinary fuzzy implication operators.

1. Introduction

Fuzzy systems mimicking human intelligence are built with the hope to replace human expertise in various cases (e.g., troubleshooting operations, etc.). Such systems achieve their goal by introducing partial set membership, new set operations, and a new logic. In order to make things clear let us give a short illustrative example. Assume that we ask Alice to tell us how hot it was today. Most probably, she will give us an answer of the form "it was really hot," "it was somewhat hot," "it was rather cold," etc., but she will not be able to tell us what was the highest temperature (unless she has consulted a thermometer...). The descriptive responses can be modelled with fuzzy set theory (probably one may employ other tools, but we are interested here in fuzzy set theory only). In addition, these descriptive answers are a typical example that can be used to fully justify the claim that classical logic is poor modeling tool. Thus, one may say that fuzzy logic can be used to *implement* human inexact reasoning in various instances. Whether this is generally possible is an open philosophical and scientific problem (e.g., see [3] for a thorough discussion of this issue). On the other hand, its should be clear that one can implement some aspects of human intelligence to mechanically solve some relatively simple problems.

Fuzzy systems make extensive use of if-then rules, also known as Horn clauses. Classically, any if-then rule is modelled by a classical implication. For example, a rule of the form "if X then Y" is modelled by the implication $X \Rightarrow Y$. Fuzzy if-then rules

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are modelled by fuzzy implications. Unlike classical logic where implication is a unique operation, in fuzzy logic there is no unique fuzzy implication operator. On the contrary, one can easily define a new fuzzy implication operator.

Intuitionistic fuzzy set (IFS) theory is a moderate extension of fuzzy set theory, which was developed by one of the present authors (see [1] for a thorough presentation of the theory of IFS). In particular, it introduces an independent non-membership degree in addition to the ordinary membership degree for each element of an intuitionistic fuzzy subset. More specifically, an intuitionistic fuzzy set is a triplet (X, μ, ν), where $\mathrm{I} = [0, 1]$, $\mu : X \rightarrow \mathrm{I}$ is a function called the *membership* function, and $\nu : X \rightarrow \mathrm{I}$ is another function called the *non-membership* function. Moreover, for all $x \in X$ it must hold that $0 \leq \mu(x) + \nu(x) \leq 1$.

Intuitionistic fuzzy set theory and its associated logic seem, at least in the authors eyes, to be well-suited to replace the use of ordinary fuzzy set theory in fuzzy systems. This paper is first attempt to define a number of new intuitionistic fuzzy implication operators and examine their properties. The new operators are actually a redefinition in the framework of IFS theory of fuzzy implication operators resented in [2].

2. Intuitionistic Fuzzy Implication Operators

In general, a fuzzy implication operator is a function $I : [0, 1] \times [0, 1] \rightarrow [0,]$, which yields the truth value of the fuzzy logical expression $p \rightarrow q$, where p and q are fuzzy propositions with truth values a and b, respectively. The truth value of $p \rightarrow q$ is denoted by $I(a, b)$.

In the almost classical textbook by Klir and Yuan [2], a good number of fuzzy implication operators are presented. Some of them, in particular those that will be the subject of our discussion below, are given in Table 1. These implication operators satisfy some of the following axioms (see [2]):

Axiom 1 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z))$.

Axiom 2 $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y))$.

Axiom 3 $(\forall y)(I(0, y) = 1)$.

Axiom 4 $(\forall y)(I(1, y) = y)$.

Axiom 5 $(\forall x)(I(x, x) = 1)$.

Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.

Axiom 7 $(\forall x, y)(I(x, y) = 1$ iff $x \leq y)$.

Axiom 8 $(\forall x, y)(I(x, y) = I(\eta(y), \eta(x)))$, where η is a negation operator.

Axiom 9 I is a continuous function.

Table 1: List of fuzzy implications from [2].

Name	Form of implication	Axioms
Zadeh	$\max(1-x, \min(x,y))$	1,2,3,4,9
Gaines-Rescher	$\begin{cases} 1, \text{if } x \leq y \\ 0, \text{if } x > y \end{cases}$	1,2,3,4$^-$,5,6,7,8
Gödel	$\begin{cases} 1, \text{if } x \leq y \\ y, \text{if } x > y \end{cases}$	1,2,3,4,5,6,7
Kleene-Dienes	$\max(1-x, y)$	1,2,3,4,6,8,9
Lukasiewicz	$\min(1, 1-x+y)$	1,2,3,4,5,6,7,8,9
Reichenbach	$1-x+xy$	1,2,3,4,6,8,9
Willmott	$\min(\max(1-x,y), \max(x,1-x), \max(y,1-y))$	4,6,8,9
Wu	$\begin{cases} 1, \text{if } x \leq y \\ \min(1-x,y), \text{if } x > y \end{cases}$	1,2,3,5,7,8
Klir and Yuan 1	$1-x+x^2y$	2,3,4,9
Klir and Yuan 2	$\begin{cases} y, \text{if } x = 1 \\ 1-x, \text{if } x \neq 1, y \neq 1 \\ 1, \text{if } x \neq 1, y = 1 \end{cases}$	2,3$^+$,4

We must note that in Table 11.1 from [2] there are some misprints. For example, Axiom 4 is not valid for the Gaines-Rescher implication, because for $y > 0$:

$$I(1,y) = y > 0.$$

However, Axiom 3 is satisfied by the second Klir-Yuan implication operator, since for $x = 0$:

$$I(0,y) = 1$$

when $y \neq 1$ and $y = 1$. These two axioms are marked in the table above with the $^-$ and 3$^+$ marks, respectively.

Now, we shall provide an interpretation of the implication operators presented in the Table above in the theory of intuitionistic fuzzy sets. In this setting, if x is a proposition, its truth-value is the ordered couple $V(x) = \langle a, b\rangle$ so that $a, b, a+b \in [0,1]$, where a and b are the degrees of validity and of non-validity of x.

Bellow we shall assume that for the three variables x, y and z above, the equalities $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ hold. The form of the corresponding intuitionistic fuzzy implication operators are given in Table 2. We should note that not all axioms are satisfied by them. Those that are satisfied are presented in the third column.

For the needs of the discussion that follows we need to define the notion of Intuitionistic Fuzzy Tautology (IFT) (see [1]):

Definition 2.1 The intuitionistic fuzzy proposition x is an IFT if and only if $a \geq b$.

Table 2: List of intuitionistic fuzzy implications

Name	Form of implication	Axioms
Zadeh	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$	2, 3, 4, 5*, 7*, 9
Gaines-Rescher	$\langle 1 - \mathrm{sg}(a - c), d \cdot \mathrm{sg}(a - c) \rangle$	1, 2, 3, 5
Gödel	$\langle 1 - (1 - c) \cdot \mathrm{sg}(a - c), d \cdot \mathrm{sg}(a - c) \rangle$	1, 2, 3, 4, 5, 7*
Kleene-Dienes	$\langle \max(b, c), \min(a, d) \rangle$	1, 2, 3, 4, 5*, 6, 8, 9
Lukasiewicz	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$	1,2,3,4,5*,8,9
Reichenbach	$\langle b + ac, ad \rangle$	2, 3, 4, 5*, 9
Willmott	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d)) \rangle$	3*, 4*, 5*, 8, 9
Wu	$\langle 1 - (1 - \min(b, c)) \cdot \mathrm{sg}(a - c),$ $\max(a, d) \cdot \mathrm{sg}(a - c) \cdot \mathrm{sg}(d - b) \rangle$	1, 2, 3, 5
Klir and Yuan 1	$\langle b + a^2 c, ab + a^2 d \rangle$	2, 3, 4, 5*
Klir and Yuan 2	$\langle c \cdot \overline{\mathrm{sg}}(1 - c) + \mathrm{sg}(1 - a) \cdot (\overline{\mathrm{sg}}(c - 1) +$ $b \cdot \mathrm{sg}(1 - c)),$ $d \cdot \overline{\mathrm{sg}}(1 - a) + a \cdot \mathrm{sg}(1 - a) \cdot \mathrm{sg}(1 - c) \rangle$	2, 3, 4

The intuitionistic fuzzy implication operators that satisfy an axiom in the sense of IFT, are marked with an asterisk ("*"). In certain cases, we need to use the functions sg and $\overline{\mathrm{sg}}$, which are defined as follows:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\overline{\mathrm{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

In ordinary intuitionistic fuzzy logic the negation of variable x is $\eta(x)$ such that $V(\eta(x)) = \langle b, a \rangle$. Also,

$$x \leq y \text{ if and only if } a \leq c \text{ and } b \geq d.$$

First, we must prove the correctness of the new definitions. For example, the second Klir-Yuan implication operator is correct, because for all $a, b, c, d \in [0, 1]$, such that $a + b \leq 1$ and $c + d \leq 1$ it holds that, if

$$Z \equiv c \cdot \overline{\mathrm{sg}}(a-1) + \mathrm{sg}(1-a) \cdot (\overline{\mathrm{sg}}(c-1) + b \cdot \mathrm{sg}(1-c)) + d \cdot \overline{\mathrm{sg}}(a-1) + a \cdot \mathrm{sg}(1-a) \cdot \mathrm{sg}(1-c)$$

then, obviously, $Z \geq 0$ and

1 if $a = 1$, then $Z = c + d \leq 1$,

2 if $a < 0$, then

$$Z = \overline{\mathrm{sg}}(c - 1) + b \cdot \mathrm{sg}(1 - c) + a \cdot \mathrm{sg}(1 - c).$$

(a) if $c = 1$, then $Z = 1$.
(b) if $c = 1$, then $Z = a + b \leq 1$.

Therefore, in all cases the pair satisfies the conditions for intuitionistic fuzziness. One of the longest (but trivial) checks is this for Willmott's implication operator. All checks are similar to the above one.

Once we have established the correctness of the definition of each operator, we need to check the validity of the aforementioned properties, which can be done in a similar manner. For instance, in order to check the validity of Axiom 5 for the first Klir-Yuan implication operator, we note that

$$I(\langle a, b\rangle, \langle a, b\rangle >) = \langle b + a^3, ab + a^2 b\rangle$$

Now,

$$Z \equiv b + a^3 - ab - a^2 b = a^2(a - b) + b(1 - a).$$

If $a \geq b$, then, obviously, $Z \geq 0$. If $a < b$, then

$$Z = b(1 - a^2) - a(b - a^2) \geq b(1 - a^2) - a(1 - a^2) = (b - a)(1 - a^2) \geq 0,$$

that is, $I(\langle a, b\rangle, \langle a, b\rangle)$ is an IFT.

3. Conclusions

We have provided definitions of classical fuzzy implication operators in the framework of the theory of intuitionistic fuzzy set theory. We are convinced that these new implication operators are an essential tool for the development of real intuitionistic fuzzy systems. In addition, the development of other similar operators will pave the road for the development of intuitionistic fuzzy systems.

References

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